EFT introduction

1. Ideas.
   In order to design a quantum field theory to describe a physical system, the following questions are important:
   1. Fields: Determine the relevant degrees of freedom.
   3. Power counting: Expansion parameters, what is the leading order description?

0. In EFT, the power counting is fundamental.
0. The key principle of EFT is that to describe the physics at some scale \( m \), we do not need to know the detailed dynamics of what is going on at the energy scales \( \Lambda > > m \).
   Such as the hydrogen ground state binding energy is:
   \[ Z_0 = \frac{1}{2} M e \alpha^2 (H) \left( \frac{m_e^2}{m_h^2} \right), \quad \frac{m_e^2}{m_h^2} \sim 10^{-8} \]
   So the correction from \( b \)-quarks is just a tiny perturbation.
   We don't have to learn about it to describe hydrogen.

We also have
1. Insensitive to quarks in the proton since \( m_e << \text{proton size} \sim 200 \text{MeV}. \) So protons rather than quarks are the right degrees of freedom.
2. Insensitive to the proton mass since \( m_e << m_p \sim 1 \text{GeV}. \) So the proton acts as a static source of e-charge.
3. A nonrelativistic Lagrangian $\mathcal{L}$ for $e^-$ suffices since $\text{med} \ll c m, \text{ve} \approx 1 \text{PeV}/\text{MeV} \ll c$.

The typical momenta in the bound state are $\text{med}$ and typical energies are $\text{En med}$.

In general, EFT’s are used in two distinct ways:
- top-down and bottom-up;
  i) Top-down: High energy $\rightarrow$ Low energy simpler theory.
    Integrate out (remove) heavier particles and match into
    a low energy theory, $\mathcal{L}_{\text{high}} \rightarrow \mathcal{L}_{\text{low}}$

    $\mathcal{L}_{\text{high}}$ and $\mathcal{L}_{\text{low}}$ will agree in the infrared (IR)
    but will differ in the ultraviolet (UV)
    The desired precision will tell us how far to go
    with $N$ the sum $n$.

  ii) Bottom-up: The underlying theory is unknown, we construct
    the EFT without reference to any other theory.
    Construct $\mathcal{L}_{\text{full}}$ by writing down the most general
    set of possible interactions consistent with all symmetries,
    using fields for the relevant degrees of freedom.

    Couplings are unknown but can be fixed fit to experimental
    or numerical data.

    fit $N$. 
The $\Sigma$ expansion is in powers, but there are also logs. Renormalization of $\Sigma_{\text{low}}$ allows us to sum large logs
$$\ln \left( \frac{m_1}{m_2} \right) \sim \left( m_2 < m_1 \right)$$

2. SM as an EFT.

SM of particle and nuclear physics can be seen as a bottom-up EFT with $\Sigma_{\text{low}} = \Sigma_{\text{low}}^{(0)} = \Sigma_{\text{low}}^{(0)} + \Sigma_{\text{low}}^{(1)} + \Sigma_{\text{low}}^{(2)} + \ldots$

The 0th order is gauge symmetry: SU(3)$_c$ x SU(2)$_W$ x U(1)$_Y$

Gauge bosons: gluons $A^a_m = 8$
Weak bosons $W^a_m = 3$
$Z^0$ bosons $B^a_m = 1$

What is $f^{(1)}$? First review some 0th order term.

$f^{(0)} = f_{\text{gauge}} + f_{\text{fermion}} + f_{\text{Higgs}} + f_{\text{top}}$

$f_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$

$f_{\text{fermion}} = \sum_i \overline{\psi}_i \gamma^\mu \gamma^5 \psi_i + \sum_i \overline{\psi}_i \gamma^\mu \gamma^5 \psi_i$

$f_{\text{Higgs}} = \frac{1}{2} m^2 H^2$

$f_{\text{top}} = i \sum \overline{u} \gamma^\mu \gamma^5 G \gamma^\mu u + \sum \overline{u} \gamma^\mu \gamma^5 G \gamma^\mu u$

The power counting for the SM as an EFT must be based on a new mass scale at the higher energy $\Lambda_{\text{new}}$. The expansion parameter should be a mass ratio $\epsilon = \frac{m_{\text{SM}}}{\Lambda_{\text{new}}}$, $m_{\text{SM}}$ is the SM particle mass ($m_W, m_Z, \ldots$)
Renormalizability in the context of an EFT:

i) Traditional definition: A theory is renormalizable if at any order of perturbation, divergences from loop integrals can be absorbed into a finite set of parameters.

ii) EFT Definition: A theory must be renormalizable order by order in its expansion parameters.

This allows for an infinite number of parameters, but only a finite number at any order in $\varepsilon$.

If our $\phi^4$ is traditionally renormalizable, it does not contain any direct information on $\Lambda$.

From the example to see how mass dimension determines power counting.

$$S[\phi] = \int d^d x \left( \frac{1}{2} \partial^2 \phi \partial^2 \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - \frac{\Gamma}{6!} \phi^6 \right)$$

$[S[\phi]] = 0, \quad [x] = 1, \quad [\phi] = \frac{d-2}{2}, \quad [m^2] = 2, \quad [\lambda] = 4-d$ and $[\Gamma] = 6-2d$.

Suppose we want to study $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ at large distance $x^i = s x'^i$ ($s \to \infty$ while keeping $x'^i$ fixed). Then to normalize the kinetic term one can redefine the large distance scalar field by

$$\phi'(x') = s^{\frac{d-2}{2}} \phi(x)$$

$$S'[\phi'] = \int d^d x' \left( \frac{1}{2} \partial^2 \phi' \partial^2 \phi' - \frac{1}{2} m^2 s^2 \phi'^2 - \frac{\lambda}{4!} s^{4-d} \phi'^4 \right)$$

$$\langle \phi'(x'_1) \cdots \phi'(x'_n) \rangle = s^{\frac{n(2-d)}{2}} \langle \phi'(x'_1) \cdots \phi'(x'_n) \rangle$$
For $d=4$, as $s \to \infty$, $m^2$ becoming more important

$$-\frac{1}{2} m^2 S^2 \phi^{'2}$$

$x$ being equally important

$$-\frac{n}{4!} S^4 \phi^{4'} = -\frac{n}{4!} \phi^{'4}$$

$I$ becoming less important

$$-\frac{7}{6!} S^6 \phi^{6'} = -\frac{7}{6!} S^6 \phi^{6'}$$

So, the operator $\phi^2$ is relevant since

$$[\phi^2] < d \quad \text{and} \quad [m^2] > 0$$

the operator $\phi^4$ is marginal since its mass dimension is

$$[\phi^4] = d \quad \text{and} \quad [m^2] > 0$$

the $\phi^6$ operator $\phi^6$ is irrelevant since

$$[\phi^6] > d \quad \text{and} \quad [m^2] > 0$$

Large distance means small momenta, so the energy scale decreases. If $m$ is the mass of a particle in a theory at a high energy scale $\Lambda \gg m$, then the $\phi^2$ operator is a small perturbation and to some extent can be neglected. Yet in the low energy scale $\Lambda \ll m$, this term represents some non-perturbative description. If $m \Lambda_{\text{new}}$ is the mass of an unknown particle for a theory at a low energy scale $\Lambda \ll \Lambda_{\text{new}}$, then $m^2 \Lambda_{\text{new}}$, $\Lambda^6 \Lambda_{\text{new}}$ and $\Lambda^2 \Lambda_{\text{new}}^2$. 
Since 2FT looks forward the IR (infrared) of the underlying theory, the mass term of the heavy particle will not be included. The $\phi^4$ and $\phi^6$ terms are included and can usually be integrated out, leaving an 2FT that contains only light degrees of freedom.

Set $m=0$, the traditional renormalization: ($d=4$ and cut-off $\Lambda$)

$$ \mathcal{L}_0 = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{\lambda}{4!} \phi^4 - \frac{\gamma}{6!} \phi^6 $$

$$ K = \lambda \int \frac{dk}{(k^2-m^2+i\epsilon)(\left(1+c^2k^2\right)^2-m^2+i\epsilon)} \sim \frac{1}{\ln \Lambda} $$

It renormalizes $\times \Lambda^2 \phi^4$

$\sim \pi \ln \Lambda$ divergences renormalizes $\times \Lambda^2 \phi^6$

$\sim \pi^2 \ln \Lambda$ divergences renormalizes $\sim \phi^8$

Yet there is no $\phi^8$ term, so the theory is non-renormalizable in the traditional sense. But if $\Lambda^2 \sim \Lambda_{\text{new}}^{-2}$ is small and $p^2 \ll \Lambda_{\text{new}}^2$, the theory can be renormalized order by order in $\Lambda_{\text{new}}$. So this given scalar field theory is renormalizable up to $\Lambda_{\text{new}}^2$ ($2\phi^6$ term). To have a renormalizable 2FT up to $\Lambda_{\text{new}}^2$, one needs to add a $\phi^8$ operator. In general, to include all corrections up to $\Lambda_{\text{new}}^{-2}$ ($R \geq 0$), one has to consider all operators with mass dimension $\leq 4R$.

In the SM, all operators have mass dimension $\leq 4$. To get the $\phi^8$ correction for the SM, we can add a mass dimension
5 operator $D_5: \gamma'' = \frac{C_5}{\Lambda_{\text{new}}} D_5$, with $D = [D_5] = 5$, $C_5 \sim 1$ and $[C_5] = 0$. Since $\gamma''$ does not contain $\Lambda_{\text{new}}$, one can take $\Lambda_{\text{new}} \gg M_{\text{SU}}$.

From experimental data, $\gamma''$ gives very small corrections.

So, toward the $1R$

$$L = L_{\text{SM}} + \gamma'' + \gamma' + \ldots = (\Lambda_{\text{new}}^0) + (\Lambda_{\text{new}}^{-1}) + (\Lambda_{\text{new}}^{-2}) + \ldots$$

Assume Lorentz invariance and gauge invariance are still unbroken, then each $\gamma''$ is Lorentz invariant and SU(3) x SU(2) x U(1) invariant. These $\gamma''$ should be constructed from the same degree of freedom as $L_{\text{SM}}$. Also assume no new particles are introduced at $\Lambda$. Then there should be new physics from those corrections.

For example, $\gamma'' = \frac{C_5}{\Lambda_{\text{new}}} E_{ij} \bar{L}_L^i H_j E^e L_L^k H^e$ is the only $D = 5$ operator consistent with symmetry. Higgs doublet $H = (h^+)$ and the $2$ lepton doublet $L_L = (\nu_L, \ell_L)$. Set $H = (0, 0)$ gives the Majorana mass term $\frac{1}{2} m_{\nu} \bar{\nu}_a \nu_a + h.c.$ with $m_{\nu} = \frac{C_5 g^2}{\Lambda_{\text{new}}}$. From experimental data $m_{\nu} \leq 0.5$ eV, so $\Lambda_{\text{new}} \geq 6 \times 10^{-4} \text{ GeV}$ ($C_5 \sim 1$).

When enumerating these operators, the classical equations of motion derived from $\gamma''$ can be used to reduce the number of operators, this is known as the integrating out at tree level.

For example:

$$L = \frac{1}{2} \partial^2 \phi \partial^2 \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 + \eta_2 \phi^6 + \eta_2 \phi^8 \phi + O(\phi^2)$$
From E.O.M. \[ \Box \phi + m^2 \phi + 4 \pi \phi^3 + O(\eta) = 0 \]
or by making a field redefinition, \[ \phi \rightarrow \phi + \eta \varphi^3 \] the new \[ \phi \text{ Lagrangian is:} \]
\[ \mathcal{L}' = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \pi' \phi^4 + \eta \varphi^3 \phi^6 + O(\eta^3) \]

Two theorems:

i) **Representation Independence Theorem:**
Consider a scalar field theory and let \( \phi = \chi F(\chi) \) with \( F(0) = 1 \). Calculations of observables with \( \mathcal{L}(\phi) \) and quantized field \( \phi \) give the same results as with \( \mathcal{L}'(\chi) = \mathcal{L}(\chi F(\chi)) \) and quantized field \( \chi \).

ii) **Generalized theorem:** Field redefinitions that preserve symmetry and have the same 1-particle states allow classical equations of motion to be used to simplify a local \( \mathcal{L} \) Lagrangian without changing observables.

For example, for a complex scalar field \( \phi \), starting from
\[ \mathcal{L}_{\text{FT}} = \sum_n \eta^n \mathcal{L}^{(n)} \]
consider removing a general first order term
\[ \frac{1}{2} [T[4] D^2 \phi \text{ from } \mathcal{L}^{(n)} \text{ that preserves symmetries, with } T[4] \text{ being a local function of various fields } \eta. \]

\[ \mathcal{Z}[J] = \int D\phi \exp \left[ i \int dx \left( \mathcal{L}^{(0)} + \eta \left( \frac{\partial}{\partial \phi} \mathcal{L}^{(1)} - T D^2 \right) + J \right) \phi \right. \]
\[ + \sum_{i \in \mathcal{J}} J_i \eta_i \phi + O(\eta^3) \]

Removing \( \frac{1}{2} [T[4] D^2 \phi \text{ is relevant to redefining the field } \phi \rightarrow \phi + \eta \phi. \]
\[ Z[J] = \int \frac{d^4 \phi}{\phi^{\star \prime}} \exp \left( i \int d^4 x \left( J_0 + \frac{1}{2} \eta T \left( \frac{\delta^2 F_0}{\delta \phi^{\star \prime}} - \frac{1}{2} \delta \phi^{\star \prime} \right) \right) + \frac{1}{2} \eta T \frac{\delta^2 \phi}{\delta \phi^{\star \prime}} + \frac{1}{2} \eta T J_0 \phi + \frac{1}{2} \eta T \phi + \frac{1}{2} \eta T \right) + O(\eta^3) \]

In it there are 3 changes, the Lagrangian, the Jacobian and the source term \( J_0 \). But from this Generalized theorem, \( J_0 \) without changing the S-matrix, we can remove the change in Jacobian and the source, so just change \( J_0 \) to 0.