E F T introduction

1. Ideas.

In order to design a quantum field theory to describe a physical system, the following questions are important,

1. Fields: Determine the relevant degrees of freedom
2. Symmetries. What interactions? Are there broken symmetries
3. Power counting: Expansion parameters, what is the leading order description!

① In EFT, the power counting is fundamental

② The key principle of EFT is that to describe the physics at some scale m, we do not need to know the detailed dynamics of what is going on at the energy scales $\Lambda_L \gg m$.

Such as the hydrogen ground state binding energy is:

$$E_0 = \frac{1}{2} m_e \alpha^2 \left(1 + O\left(\frac{m_e^2}{m_b^2}\right)\right), \quad \frac{m_e^2}{m_b^2} \sim 10^{-8}$$

So the correction from b-quarks is just a tiny perturbation, we don't have to learn about it to describe hydrogen.

We also have

1. Insensitive to quarks in the proton since $m_e \alpha \ll (\text{proton size})^{-1} \sim 200\,\text{MeV}$. So protons rather than quarks are the right degrees of freedom.
2. Insensitive to the proton mass since $m_e \alpha \ll m_p \sim 1\,\text{GeV}$. So the proton acts as a static source of e-charge.

3. A nonrelativistic Lagrangian $\mathcal{L}$ ~~is~~ for e^- ~~is~~ suffices ~~and~~ since $m_e \alpha \ll m_e$. $v_e = |\vec{p}_e|/m_e \sim \alpha \ll 1$

The typical momenta in the bound state are $\vec{p} \sim m_e \alpha$ and typical energies are $E \sim m_e \alpha^2$.

In general, EFT's are used in two distict ways: top-down and bottom-up:

i) Top-down: High energy $\longrightarrow$ Low energy simpler theory.

Integrate out (remove) heavier particles and match onto a low energy theory. $\mathcal{L}_{high} \approx \sum_n \mathcal{L}_{low}^{(n)}$

$\mathcal{L}_{high}$ and $\mathcal{L}_{low}$ will agree ~~as~~ in the infrared (IR) but will differ in the ultraviolet (UV)

The desired precision will tell us how far to go with ~~n~~ the sum n.

ii) Bottom-up: The underlying theory is unknown, we construct the EFT without reference to any other theory.

- Construct $\sum_n \mathcal{L}^{(n)}$ by writing down the most general set of possible interactions consistent with all symmetries, using fields for the relevant degrees of freedom.
- Couplings are unknown but can be ~~fixe~~ fit to experimental or numerical data.
- fit n.

The Σ_n expansion is in powers, but there are also logs. Renormalization of $\mathcal{L}_{low}^{(n)}$ allows us to sum large logs $\ln\left(\frac{m_1}{m_2}\right) \sim 1$ $(m_2 << m_1)$

2. SM as an EFT.

SM of particle and nuclear physics can bee seen as a bottom up EFT w/ $\sum_n \mathcal{L}_{low}^{(n)} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \ldots$

The 0'th order: gauge symmetry: $SU(3)_{color} \times SU(2)_{weak} \times U(1)_Y$

gauge bosons: gluons A_μ^A : 8

weak bosons W_μ^a : 3

$U(1)$ bosons B_μ : 1

what is $\mathcal{L}^{(1)}$? First review some 0'th order term.

$$\mathcal{L}^{(0)} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_{Higgs} + \mathcal{L}_{\nu_R}$$

$$\mathcal{L}_{gauge} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}G^{A\mu\nu}G^A_{\mu\nu}$$

$$\mathcal{L}_{fermion} = \sum_{\psi_L} \bar{\psi}_L i \not{D} \psi_L + \sum_{\psi_R} \bar{\psi}_R i \not{D} \psi_R$$

$$iD_\mu = i\partial_\mu + g_1 B_\mu Y + g_2 W_\mu^a T^a + g_3 A_\mu^A T^A$$

The power counting for the SM as an EFT must be based on a new mass scale at the higher energy Λ_{new}. The expansion parameter should be a mass ration $\epsilon = \frac{m_{SM}}{\Lambda_{new}}$, m_{SM} is the SM particle mass $(m_W, m_Z \ldots)$

Renormalizability in the context of an EFT:

i) Traditional definition: A theory is renormalizable if at any order of perturbation, divergences from loop integrals can be absorbed into a finite set of parameters.

ii) EFT Definition: A theory must be renormalizable order by order in its expansion parameters.

This allows for an infinite number of parameters, but only a finite number at any order in ε.

If an $\mathcal{L}^{(0)}$ is traditionally renormalizable, it does not contain any direct information on Λ_{new}.

From the example to see how mass dimension determines power counting.

$$S[\phi] = \int d^d x \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6\right)$$

$[S[\phi]]=0$, $[x]=-1$. $\therefore$ $[\phi]=\frac{d-2}{2}$, $[m^2]=2$, $[\lambda]=4-d$ and $[\tau]=6-2d$.

Suppose we want to study $\langle\phi(x_1)\cdots\phi(x_n)\rangle$ at large distance $x^\mu = s x'^\mu$ ($s\to\infty$ while keeping x'^μ fixed). Then to normalize the kinetic term one can redefine the large distance scalar field by $\phi'(x') = s^{\frac{d-2}{2}}\phi(x)$

$$\therefore S'[\phi'] = \int d^d x' \left(\frac{1}{2}\partial^\mu\phi'\partial_\mu\phi' - \frac{1}{2}m^2 s^2\phi'^2 - \frac{\lambda}{4!}s^{4-d}\phi'^4 - \frac{\tau}{6!}s^{6-2d}\phi'^6\right)$$

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = s^{\frac{n(2-d)}{2}}\langle\phi'(x_1')\cdots\phi'(x_n')\rangle$$

For $d=4$, as $s \to \infty$, m^2 becoming more important

$$-\frac{1}{2} m^2 s^2 \phi'^2$$

λ being equally important

$$-\frac{\lambda}{4!} s^{4-d} \phi'^4 = -\frac{\lambda}{4!} \phi'^4$$

τ becoming less important

$$-\frac{\tau}{6!} s^{6-2d} \phi'^6 = -\frac{\tau}{6!} s^{-2} \phi'^6$$

So, the operator ϕ^2 is relevant since

$[\phi^2] < d$ and

$[m^2] > 0$

the operator ϕ^4 is marginal since its mass dimension is ~~$[\phi^4] = d$~~ $[\phi^4] = d$ and

$[\lambda] = 0$

the operator ϕ^6 is irrelevant since

$[\phi^6] > d$ and

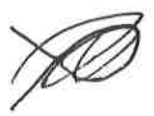
$[\tau] < 0$

Large distance means small momenta, so the energy scale decrease. If m is the mass of a particle in a theory at a high energy scale $\Lambda_E \gg m$, then the ϕ^2 operator is a small perturbation and to some extent can be neglected. Yet in the low energy scale $\Lambda_E \ll m$, this term represents some non-perturbative description. If $m \sim \Lambda_{new}$ is the mass of an unknown particle for a theory at a low energy scale $\Lambda_E \ll \Lambda_{new}$ then $m^2 \sim \Lambda^2_{new}$, $\lambda \sim \Lambda^0_{new}$ and $\tau \sim \Lambda^{-2}_{new}$.

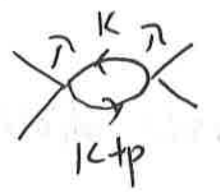
Since EFT looks forward the IR (infrared) of the underlying theory, the mass term of the heavy particle will not be included. The ϕ^4 and ϕ^6 terms are included and can usually be integrated out, leaving an EFT that contains only light degrees of freedom.

the traditional renormalization : ($d=4$ and cut-off Λ)

Set $m=0$, the

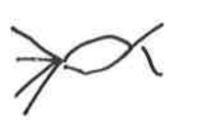


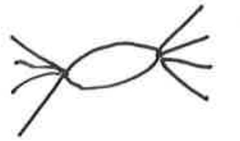
$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{\lambda}{4!}\phi^4 - \frac{\tau}{6!}\phi^6$$



$$\sim \lambda^2 \int \frac{d^d k}{(k^2-m^2+i0)((k+p)^2-m^2+i0)} \sim \int_0^\Lambda \frac{d^d k}{k^4} \sim \ln\Lambda$$

it renormalizes  $\lambda\phi^4$

 $\sim \lambda\tau \ln\Lambda$ divergence renormalizes  $\tau\phi^6$

 $\sim \tau^2 \ln\Lambda$ divorgence renormalizes  $\sim \phi^8$

Yet there is no ϕ^8 term, so the theory is non-renormalizable in the traditional sense. But if $\tau \sim \Lambda_{new}^{-2}$ is small and $p^2\tau << 1$, the theory can be renormalized order by order in Λ_{new}. So this given scalar field theory is renormalizable up to Λ^{-2}_{new} ($\tau\phi^6$ term). To have a renormalizable EFT up to Λ^{-4}_{new}, one needs to add a ϕ^8 operator. In general, to include all corrections up to Λ^{-r}_{new} ($r \geq 0$), one has to consider all operators with mass dimension $\leq d+r$.

In the SM $\mathcal{L}^{(4)}$, all operators have mass dimension ≤ 4. To get the $\mathcal{L}^{(5)}$ correction for the SM, we can add a mass dimension

5 operator $\mathcal{O}_5$: $\mathcal{L}^{(1)} = \frac{c_5}{\Lambda_{new}} \mathcal{O}_5$, with $D = [\mathcal{O}_5] = 5$, $c_5 \sim 1$ and $[c_5] = 0$. Since $\mathcal{L}^{(0)}$ doesn't contain Λ_{new}, one can take $\Lambda_{new} \gg m_{SM}$.

From experimental data, $\mathcal{L}^{(1)}$ gives very small corrections.

So, toward the IR

$$\mathcal{L} = \mathcal{L}^{SM} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots = (\sim \Lambda_{new}^{0}) + (\sim \Lambda_{new}^{-1}) + (\sim \Lambda_{new}^{-2}) + \cdots$$

Assume Lorentz invariance and gauge invariance are still unbroken, then each $\mathcal{L}^{(n)}$ is Lorentz invariant and $SU(3) \times SU(2) \times U(1)$ invariant. These $\mathcal{L}^{(n)}$ should be constructed from the same degree of freedom as $\mathcal{L}^{SM}$. Also assume no new particles are introduced at Λ. Then there should be new physics from those corrections.

For example, $\mathcal{L}^{(1)} = \frac{c_5}{\Lambda_{new}} \varepsilon_{ij} \bar{L}_L^{ci} H^j \varepsilon_{kl} L_L^k H^l$ is the only $D=5$ operator consistent with symmetry. Higgs doublet $H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ and lepton doublet $L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. Set $H = \begin{pmatrix} 0 \\ v \end{pmatrix}$ gives the Majorana mass term $-\frac{1}{2} m_\nu \varepsilon_{ab} \nu_L^a \nu_L^b + h.c.$ with $m_\nu = \frac{c_5 v^2}{2\Lambda_{new}}$. From experimental data $m_\nu \lesssim 0.5$ eV, so $\Lambda_{new} \gtrsim 6 \times 10^{14}$ GeV $(c_5 \sim 1)$.

When enumerating these operators, the classical equations of motion derived from $\mathcal{L}^{SM}$ can be used to reduce the number of operators, this is known as the integrating out at tree level.

For example: $\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 + \eta g_1 \phi^6 + \eta g_2 \phi^3 \Box \phi + O(\eta^2)$

From E.O.M: $\Box\phi + m^2\phi + 4\lambda\phi^3 + O(\eta) = 0$
or by making a field redefinition $\phi \to \phi + \eta g_2 \phi^3$, the new Lagrangian is:

$$\mathcal{L}' = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda'\phi^4 + \eta g_1'\phi^6 + O(\eta^2)$$

Two theorems:

i) Representation Independence Theorem:
Consider a scalar field theory and let $\phi = \chi F(\chi)$ with $F(0)=1$. Calculations of observables with $\mathcal{L}(\phi)$ and quantized field ϕ give the same results as with $\mathcal{L}'(\chi) = \mathcal{L}(\chi F(\chi))$ and quantized field χ.

ii) Generalized theorem: Field redefinitions that preserve symmetries and have the same 1-particle states allow classical equations of motion to be used to simplify a local EFT Lagrangian without changing observables.

For example, for a complex scalar field ϕ, starting from $\mathcal{L}_{EFT} = \Sigma_n \eta^n \mathcal{L}^{(n)}$, consider removing a general first order term $\frac{1}{2}\eta T[\psi] D^2\phi$ from $\mathcal{L}^{(1)}$ that preserves symmetries, with $T[\psi]$ being a local function of various fields ψ.

$$Z[J] = \int \prod_i D\psi_i \exp\left(i\int d^4x \left(\mathcal{L}^{(0)} + \eta(\mathcal{L}^{(1)} - TD^2\phi) + \eta TD^2\psi + \sum_k J_k\psi_k + O(\eta^2)\right)\right)$$

Removing $\frac{1}{2}\eta T[\psi]D^2\phi$ is relevant to redefining the field $\phi^* = \phi^* + \eta T$:

$$Z[J] = \int \prod_i D\psi_i' \frac{\delta\phi^*}{\delta\phi'^*} \exp\Big(i\int d^dx \Big(\mathcal{L}^{(0)} + \frac{1}{2}\eta T\Big(\frac{\delta\mathcal{L}^{(0)}}{\delta\phi^*} - \partial_\mu \frac{\delta\mathcal{L}^{(0)}}{\delta\partial_\mu\phi^*}\Big)$$

$$+\eta\Big(\mathcal{L}^{(1)} - \frac{1}{2}TD^2\phi'\Big) + \frac{1}{2}\eta TD^2\phi' + \sum_k J_k\psi_k' + \frac{1}{2}J_{\phi^*}\eta T$$

$$+O(\eta^2)\Big)$$

In it there are 3 changes, the Lagrangian, the Jacobian and the source term J_{ϕ^*}. But from this Generalized theorem, we can remove the without changing the S-matrix, we can remove the change in Jacobian and the source, so just change the $\mathcal{L}$.