Introduction to Spacetime algebra (STA)

STA is a special (4D) case of geometric algebra (GA). GA is a mathematical language known long before it has an application in physics. A new kind of product among vectors called "geometric product" was introduced in this algebra which is defined between two vectors, \( \mathbf{A} \) and \( \mathbf{B} \) as

\[
\mathbf{A} \mathbf{B} = \mathbf{A} \mathbf{B} + \mathbf{A} \wedge \mathbf{B}
\]

\[
\text{geometric inner outer product product product}
\]

The inner product is our usual dot product giving a scalar. However, the outer product is something new. Geometrically, it is interpreted as an oriented plane having \( \mathbf{A} \) and \( \mathbf{B} \) as two sides of this plane:

\[
\mathbf{A} \wedge \mathbf{B} \leftrightarrow \frac{1}{2} [\mathbf{A} \mathbf{B} + \mathbf{B} \mathbf{A}]
\]

In this picture, it implies that \( \mathbf{A} \wedge \mathbf{A} = 0 \)

and \( \mathbf{A} \wedge \mathbf{B} = - \mathbf{B} \wedge \mathbf{A} \).

Outer product gives us a new geometric object which we will call a bivector clearly because it is obtained by forming outer product of two vectors.

For example in 2D case, let \( \mathbf{A} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \) , \( \mathbf{B} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 \) then the geometric product is

\[
\mathbf{A} \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}
\]

\[
= (a_1 b_1 + a_2 b_2) \mathbf{e}_1 \mathbf{e}_2 + \mathbf{A} \wedge \mathbf{B}.
\]

For special case when \( \mathbf{A} \mathbf{B} \) we have \( \mathbf{A} \mathbf{B} = \mathbf{A} \mathbf{B} \rightarrow \text{bivector} \)

or when \( \mathbf{A} \mathbf{B} \) we have \( \mathbf{A} \mathbf{B} = \mathbf{A} \mathbf{B} \rightarrow \text{scalar} \).

Geometric product satisfies the usual axioms of associativity and distributivity (over addition). But clearly it is not commutative.

Therefore in our 2D case we have three objects namely scalar, vectors and bivector.

\[
\begin{align*}
1 & \quad \mathbf{e}_1, \mathbf{e}_2, \quad \mathbf{e}_1 \mathbf{e}_2, \text{ or we can now write } \mathbf{e}_1 \mathbf{e}_2.
\end{align*}
\]

The associated geometric algebra is defined as a space spanned by those objects with geometric product being a fundamental operation. Hence we can write the most general element in this algebra as follow:

\[
\Psi = a + b \mathbf{e}_1 + c \mathbf{e}_2 + d \mathbf{e}_1 \mathbf{e}_2 - \text{0}
\]

which we will call a multivector. We also have a name for:

scalar component of \( \Psi \rightarrow \langle \Psi \rangle_0 = \text{grade-0 component} \)

vector component of \( \Psi \rightarrow \langle \Psi \rangle_1 = \text{grade-1 component} \)

bivector component of \( \Psi \rightarrow \langle \Psi \rangle_2 = \text{grade-2 component} \)

Therefore from \( \text{0} \) we have

\[
\begin{align*}
\langle \Psi \rangle_0 &= a, \quad \langle \Psi \rangle_1 = b \mathbf{e}_1 + c \mathbf{e}_2, \quad \langle \Psi \rangle_2 = d \mathbf{e}_1 \mathbf{e}_2.
\end{align*}
\]

We call \( \Psi \) "even" when it is composed of \( \langle \Psi \rangle_0 \)'s where \( n \) is an even number and "odd" when it's an odd number. For example

\[
\Psi_{\text{even}} = a + d \mathbf{e}_1 \mathbf{e}_2, \quad \Psi_{\text{odd}} = d \mathbf{e}_1 + c \mathbf{e}_2.
\]

\( \Psi_{\text{even}} \) in the above form also form a subalgebra called even algebra which has a very interesting feature that an element in this subalgebra...
can represent a complex number where \( \hat{e}_1 \hat{e}_2 \) acts like an imaginary unit \( \text{i} \) according to

\[
(\hat{e}_1 \hat{e}_2)^2 = (\hat{e}_1 \hat{e}_2)(\hat{e}_1 \hat{e}_2) = -\hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 = -\hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_2 = -1
\]

Thus \( \hat{e}_1 \hat{e}_2 = \sqrt{-1} \) which can be interpreted as a "real" object (oriented plane) in GA language.

Therefore arbitrary element in this even subalgebra is isomorphic to a complex number that is

\[ a + b \hat{e}_1 \hat{e}_2 \leftrightarrow a + ib \]

Now let's look at its application in physics. Let's first consider in 3D. In this case we have 3 unit base vectors \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) but let us instead denote them as \( \sigma_1, \sigma_2, \sigma_3 \). We can construct many geometric objects from these vectors again by using the geometric product. Then we can obtain its associated geometric algebra as a space spanned by those objects. In this case our GA is an 8D space having the following as bases:

1. scalar: 1
2. vectors: \( \sigma_1, \sigma_2, \sigma_3 \)
3. bivectors: \( \sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_3 \sigma_1 \)
4. trivector: \( \sigma_1 \sigma_2 \sigma_3 = \text{i} \)

Here we meet new object a "trivector". It is simply interpreted as an oriented volume analogous to how we have interpreted a bivector.

However the square of this trivector basis also yields \(-1\). So we now have 4 basis having this property namely \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) and \( \sigma_5, \sigma_6 \).

Because

\[
\begin{align*}
(\sigma_1 \sigma_2)^2 &= \sigma_1 \sigma_2 \sigma_1 \sigma_2 = -\sigma_1 \sigma_2 \sigma_1 \sigma_2 = -1 \\
(\sigma_3 \sigma_4)^2 &= \sigma_3 \sigma_4 \sigma_3 \sigma_4 = -\sigma_3 \sigma_4 \sigma_3 \sigma_4 = -1 \\
(\sigma_5 \sigma_6)^2 &= \sigma_5 \sigma_6 \sigma_5 \sigma_6 = -\sigma_5 \sigma_6 \sigma_5 \sigma_6 = -1
\end{align*}
\]

Therefore arbitrary element in this even subalgebra is isomorphic to a complex number that is

\[ a + b \sigma_3 \sigma_4 \leftrightarrow a + ib \]

Then which one of these represents the imaginary unit is not clear.

The application of this 3D version of GA in physics was introduced by David Hestenes (1966) where he proposed that we can think of the Pauli matrices (used in Schrödinger-Pauli equation) as unit orthogonal vectors in 3D space. Then he proposed that a spinor is an even element of the associated GA hence allowed him to rewrite the equation in the language of GA which is coordinate-free and representation-free. Let's now look at it in details.
First let's look at an arbitrary even element in this GA which we will identify as a spinor in Pauli's theory. It has a form:

\[ \Psi_{even} = a + b \sigma_2 + c \sigma_3 + d \sigma_2 \sigma_3 - 6 \]

We can rewrite \( \sigma_3 \) as \( \sigma_3 \sigma_2 \sigma_3 = i \sigma_3 \)

and also \( \sigma_3 = \sigma_3 \sigma_2 \sigma_3 = -i \sigma_2 \)

likewise \( \sigma_2 = \sigma_2 \sigma_3 \sigma_2 = i \sigma_1 \)

and then rename \( a \rightarrow a^0 \)

\( d \rightarrow a^1 \)

\( -c \rightarrow a^2 \)

\( b \rightarrow a^3 \)

Thus \( \Psi \) can be rewritten as \( \Psi_{even} = a^0 + a^1 i \sigma_3 \)

Next we want to identify this \( \Psi \) as a spinor.

A spinor is a column matrix with two complex entries

\[ |\Psi\rangle = (\Psi_1, \Psi_2) \quad \Psi_1, \Psi_2 \in \mathbb{C} \]

which has 4 degrees of freedom.

To draw a one-to-one correspondence between \( \Psi_{even} \) and \( |\Psi\rangle \) we first write \( |\Psi\rangle \) in the form of (a)

But written in matrix form:

\[ |\Psi\rangle = \begin{pmatrix} b^0 + j a^3 \\ b^3 + j a^0 \end{pmatrix} \]

Here we extend a column matrix to a 2×2 matrix where the element \( x \) and \( y \) are out of our interest, and \( j \) is an imaginary unit.

we then find that

\[ b^0 = a^0, \quad b^3 = a^3, \quad b^1 = -a^1, \quad b^2 = -a^2 \]

Therefore an even multivector \( \Psi = a^0 + a^1 i \sigma_3 \)

is in one-to-one correspondence with the spinor

\[ |\Psi\rangle = \begin{pmatrix} a^0 + j a^3 \\ -a^2 + j a^1 \end{pmatrix} \]

Up to now we can translate spinor from the language of matrix algebra to GA.

Next we look at how operators can be translated in GA. For example let's look at spin operators first. Use spinor in (b) we obtain:

\[ \hat{a}^0 |\Psi\rangle = \begin{pmatrix} -a^2 + j a^1 \\ a^0 + j a^3 \end{pmatrix} \]

this new spinor can be directly translated to GA as the following even multivector:

\[ (-a^2 + j a^1) \quad \leftrightarrow \quad -a^2 + a^1 i \sigma_1 - a^3 i \sigma_2 + a^0 i \sigma_3 \]

then we try to rearrange this to include \( \Psi \) of the form in (b)

We then Find

\[ -a^2 + a^0 i \sigma_1 - a^3 i \sigma_2 + a^1 i \sigma_3 = -a^2 + a^0 i \sigma_1 - a^3 i \sigma_2 + a^1 i \sigma_3 \]

\[ = \sigma_1 \begin{pmatrix} -a^2 + a^0 i \sigma_1 - a^3 i \sigma_2 + a^1 i \sigma_3 \end{pmatrix} \]

\[ = \sigma_1 \begin{pmatrix} a^1 i \sigma_2 - a^0 i \sigma_3 + a^0 + a^1 i \sigma_3 \end{pmatrix} \]

\[ = \sigma_1 \begin{pmatrix} a^1 i \sigma_2 + a^0 i \sigma_3 + a^0 + a^1 i \sigma_3 \end{pmatrix} \]

\[ = \sigma_1 \begin{pmatrix} a^1 i \sigma_2 + a^0 i \sigma_3 + a^0 + a^1 i \sigma_3 \end{pmatrix} \]

\[ = \sigma_{1} a^1 \sigma_2 + a^0 i \sigma_3 + a^0 + a^1 i \sigma_3 \]

\[ = \sigma_{1} a^1 \sigma_2 + a^0 i \sigma_3 + a^0 + a^1 i \sigma_3 \]

\[ = \sigma_{1} \Psi \sigma_3 \]
This means \( \hat{\psi}(\varphi) \leftrightarrow \hat{\psi} \alpha \), and likewise for \( \hat{\sigma}_3 \) and \( \hat{\sigma}_2 \) we can write the translation of spin operator \( \hat{\sigma}_k \) to GA as

\[
\hat{\sigma}_k(\varphi) \leftrightarrow \hat{\psi}_k \alpha \quad - (6)
\]

Another term that appears in Shrödinger–Pauli eqn is \( j \psi \). However this is not simply \( j \psi \) in GA. The \( j \) term acts like an operator.

\[
j \psi = \begin{pmatrix} -a^1 - ja^2 \\ -a^3 + ja\end{pmatrix}
\]

which translates as

\[
j \psi \leftrightarrow \psi^* \sigma_3 \quad - (7)
\]

In the same way we also have to translate the operation “complex conjugate” in to our GA

\[
\psi^* = \begin{pmatrix} a^1 - ja^2 \\ a^3 - ja\end{pmatrix}
\]

This means \( \psi^* \leftrightarrow \psi^* \sigma_3 \quad - (8) \)

Now we can write Pauli equation:

\[
\frac{\partial}{\partial t} \psi = \frac{1}{2m} \left( -J^\nu - e A^\nu \right) \psi + e V \psi
\]

in the language of GA as

\[
\frac{\partial}{\partial t} \hat{\psi} = \frac{1}{2m} \left( -\mathbf{e} \cdot \mathbf{A} \right) \hat{\psi} + eV \hat{\psi}
\]

where \( \mathbf{B} = B^k \mathbf{e}_k \) is the magnetic field vector.

We will not concern about solving eqn (3) here but there are a couple of interesting features when we deal with Pauli equation in GA language. First there is no more imaginary unit \( j \). Instead we have a trivector \( i = \hat{\psi}_2 \alpha \), which is a real geometric object. Secondly eqn (5) has no reference to any frame or any matrix representation. Finally we turn operators and spinors into the multiplication of multivector which is easier to solve compared to eqn (8).

However eqn (8) & (9) is equivalent thus eqn (9) will not provide additional prediction. But there is geometrical meaning to be interpreted which might lead to more profound understanding of quantum mechanics.

The next thing we want to discuss is how to calculate expectation value once we know \( \psi \) in GA. It is easy to verify that if the inner product is a real value (which is the case for the expectation value but not true for probability amplitude) then it is equivalent to just the grade-0 component of the product of the two multivectors of interest. That is

\[
\text{Re}[\langle \psi \psi \rangle] = \langle \psi^* \rangle
\]

Here we simply use \( \psi^* = a^0 - i a^k \). But if our inner product happens to be complex then the full version will be

\[
\langle \psi \psi \rangle = \langle \psi^* \rangle - \langle \psi^* i \sigma_3 \rangle i \sigma_3
\]
For example, let's compute $\langle \psi | \hat{\sigma} | \psi \rangle$ which is always real.

$\langle \psi | \hat{\sigma} | \psi \rangle \Rightarrow \langle \psi | \hat{\sigma}_k | \psi \rangle_0$

We can write r.h.s. in more informative form as

$\Rightarrow \delta_k \cdot \langle \psi \cdot \psi \rangle$

This indicates that it is the component of the spin vector $\psi \cdot \psi$ in the $\delta_k$ direction, so that

$S = \psi \cdot \psi$ is the coordinate-free form of this vector.

Now let's look at Dirac equation. Apparently we are dealing with 4D spacetime. Analogous to the Pauli case we can think of Dirac's $\gamma$-matrices as 4 unit orthogonal vectors in our spacetime. And then we will obtain an associated GA which we will now call Spacetime Algebra (STA). Then we will identify a Dirac's spinor to an element in STA and rewrite Dirac eqn in STA's language. Then we will talk about observable in STA. We will see that in STA although it doesn't provide any new physical prediction, it provides some geometrical interpretation in every term just like in the Pauli case.

In 4D case we have 4 bases $\gamma_0, \gamma_1, \gamma_2$ and $\gamma_3$. We can combine them to create 16 objects:

- 1 scalar: 1
- 4 vectors: $\gamma_0, \gamma_1, \gamma_2, \gamma_3$
- 6 bivectors: $\gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_2 \gamma_3, \gamma_0 \gamma_1, \gamma_0 \gamma_2, \gamma_0 \gamma_3$
- 3 trivectors: $\gamma_0 \gamma_1 \gamma_2, \gamma_0 \gamma_1 \gamma_3, \gamma_0 \gamma_2 \gamma_3$
- 1 pseudoscalar: $\gamma_0 \gamma_1 \gamma_2 \gamma_3 = i$

We then identify a spinor $| \psi \rangle$ with even element of STA as follows:

$| \psi \rangle = \left( \begin{array}{c} \alpha + \beta \gamma^0 \\ \gamma^1 \alpha + \beta \gamma^2 \\ \gamma^3 \alpha + \beta \gamma^0 \\ -\gamma^2 \alpha + \beta \gamma^3 \end{array} \right) \Rightarrow \alpha + \beta \delta_k + i (\beta_0 \beta_1 \beta_2 \beta_3)$

Analogous to the Pauli case we can find

- $\phi^m | \psi \rangle \Rightarrow \gamma_m | \psi \rangle$
- $j | \psi \rangle \Rightarrow \gamma_1 | \psi \rangle$
- $m | \psi \rangle \Rightarrow -\gamma_2 | \psi \rangle$

which allows us to write the Dirac equation:

$\delta^m (j_2 - eA_m) | \psi \rangle = m | \psi \rangle$

in STA language as

$\nabla \psi \cdot \gamma_3 - eA_\psi = m | \psi \rangle$