A Brief Introduction to Resurgence, trans series & Alien calculus

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for hard Problems our first guess is to solve hopefully a simpler Problem, Perhaps with some coupling $g=0$, Then solve Perturbatively

$$O(g) = c_0 + c_1 g + c_2 g^2$$

The Physical observable O is then a series in terms of g.

- This series has been shown via standard arguments (Dyson Phys.Rev. 85 (1952)) is asymptotic and must have vanishing Radius of Curvature, $g \sim 0$

- Borel Resummation was introduced as an analytic continuation of our asymptotic series via contour integration of the Borel trans. in the complex plane.

- This leads to a trans-series

$$O(g) = \sum_{n \geq 0} c_n^{(0)} g^n + \sum_i e^{-S_i/g} \sum_{n \geq 0} c_n^{(i)} g^n$$

these cannot be captured by $g \ll 1$ physics.

Borel Resummation:

We work with $z = 1/g$ such that $g \to 0$ $z \to \infty$

Denote $\mathbb{C}[[z^{-1}]]$ - The set of all formal Power series in $1/z$

in Particular we are interested in

$$z^{-1}\mathbb{C}[[z^{-1}]] = \left\{ \sum_{n=0}^{\infty} c_n z^{-n-1} ,\ c_n \in \mathbb{C} \right\}$$

Outline :)

1.) Motivation
2.) Define the Borel transformation
3.) What are the Properties of this transformation?
4.) What type of expansion do we encounter in QFT?
5.) Inverse Borel transformation?
6.) Example
7.) Conclusion

Side note :

- This subject is emense.
- These notes should provide you with what you need in order to finish reading 0405279
- inaddition the beginning of Transseries for Beginners (G.A. Edgar) is a lovely Read which fills in some missing info.
- in addition there are some interesting works out there one i would like to try and read is, Alien Calculus and a Schwinger-Dyson equation: two point f^ with a nonperturbative mass scale. Marc Bellon Pierre Clavier

- f^ ... function
- Def^ ... Definition
- basically I use ^ to stand in for the rest of the word

Def$\hat{=}$ 1.) Define the linear op. $\mathcal{B}$ The Borel transformation

$$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$$

$$\mathcal{B} : \tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \to \hat{\Phi}(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n$$

lets uncover some properties of the Borel transformation

first clearly this op implies

$$\mathcal{B}(z^{-n-1}) = \zeta^n / n! = \zeta^n / \Gamma(n+1)$$

What about: $\mathcal{B}\left(\frac{d}{dz}\tilde{\Phi}(z)\right) \overset{?}{=} \mathcal{B}(-c_0 z^{-2} - c_1 2 z^{-3} - c_2 3 z^{-4} + \cdots)$

$$= -c_0\zeta - \frac{c_1 2 \zeta^2}{2!} - \frac{3\zeta^3}{3!}c_2 \cdots\cdots$$

$$= -\zeta\left(c_0 + c_1\zeta + \frac{c_2}{2!}\zeta^2 + \cdots\right) = -\zeta\hat{\Phi}(\zeta) \checkmark$$

$$\mathcal{B}(z\tilde{\Phi}(z)) = \mathcal{B}(c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots)$$

$= \cancel{\mathcal{B}}$

$$= c_1 + c_2\zeta + c_3\frac{\zeta^2}{2} + \cdots$$

$$= \frac{d}{d\zeta}\hat{\Phi}(\zeta) \checkmark$$

$$\mathcal{B}(\tilde{\Phi}(\lambda z)) = \mathcal{B}(c_0 z^{-1}\lambda^{-1} + c_1 z^{-2}\lambda^{-2} + c_2 z^{-3}\lambda^{-3} + \cdots)$$

$$= \lambda^{-1}\left(c_0 + c_1\zeta\lambda^{-1} + \frac{c_2\zeta^2\lambda^{-2}}{2!} + \cdots\right)$$

$$= \lambda^{-1}\hat{\Phi}(\lambda^{-1}\zeta) \checkmark$$

$$\mathcal{B}(\hat{\Phi}_1(z)\hat{\Phi}_2(z)) = \mathcal{B}\left[\left(\sum_{n=0}^{\infty} a_n z^{-n-1}\right)\times\left(\sum_{n=0}^{\infty} b_n z^{-n-1}\right)\right]$$

$$= \mathcal{B}\left[(a_0 z^{-1} + a_1 z^{-2} + a_2 z^{-3} + \cdots)(b_0 z^{-1} + b_1 z^{-2} + b_2 z^{-3} + \cdots)\right]$$

$$= \mathcal{B}\left[(a_0 b_0 z^{-2} + (a_0 b_1 + b_0 a_1) z^{-3} + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^{-4} + \cdots\right]$$

Notice coefficients can be written

$$C_n = \sum_{p+q=n-1} a_p b_q, \quad n \geq 1$$

$n=1$ $p=-q$ $\Rightarrow$ $p=q=0$, $\boxed{a_0 b_0}$

$n=2$ $p+q=1$ $\Rightarrow$ $p=0, q=1$, $q=0, p=1$ $\boxed{a_0 b_1 + b_0 a_1}$

$n=3$ $p+q=2 \Rightarrow p=0, q=2$, $p=1, q=1$, $p=2, q=0$

$\boxed{a_0 b_2 + a_1 b_1 + a_2 b_0}$

$$= \mathcal{B}\left[\sum_{n=1}^{\infty} C_n z^{-n-1}\right]$$

$$= \sum_{n=1} \frac{C_n}{n!}\zeta^n = \sum_{n=1}^{\infty} \sum_{p+q=n-1} \frac{a_p b_q}{n!}\zeta^n$$

$$= \sum_{p,q\geq 0} \frac{a_p b_q}{(p+q+1)!}\zeta^{p+q+1}$$

Note $p+q=n-1$

$\Rightarrow n = p+q+1$

Summing over n

$n=1$ $1 = p+q+1$

$0 = p+q$

$n=2$ $1 = p+q$

$n=3$ $2 = p+q$

This implies summing over $p, q \geq 0$

Next The Euler Beta fn is $B(m,n) = \dfrac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$

or

$$\frac{B(m+1,n+1)}{\Gamma(m+1)\Gamma(n+1)} = \frac{1}{\Gamma(n+m+2)} = \frac{1}{(m+n+1)!}$$

Recall $\Gamma(n) = (n-1)!$ $n \in \mathbb{Z}$

$$= \sum_{p,q\geq 0} \frac{a_p b_q}{\Gamma(p+1)\Gamma(q+1)} B(p+1,q+1)\zeta^{p+q+1}$$

Recall further $B(p+1,q+1) = \int_0^1 t^p(1-t)^q dt$

Then $\zeta^{p+q+1} B(p+1,q+1) = \int_0^1 t^p(1-t)^q dt\, \zeta^{p+q+1}$

Change variables $\zeta_1 = \zeta t \Rightarrow \zeta_{1L} = 0$, $\zeta_{1R} = \zeta$

$d\zeta_1 = \zeta dt$

$$= \int_0^\zeta \zeta_1^p \zeta^{-p} \zeta^{-q} (\zeta - \zeta_1)^q \zeta^{-1} d\zeta_1\, \zeta^{p+q+1}$$

$$= \int_0^\zeta \zeta_1^p (\zeta - \zeta_1)^q d\zeta_1$$

finally we arrive at

$$\mathcal{B}(\hat{\varphi}_1 \hat{\varphi}_2) = \sum_{p,q \geq 0} \frac{a_p b_q}{\Gamma(p+1)\Gamma(q+1)} \int_0^{\zeta} \zeta_1^p (\zeta - \zeta_1)^q \, d\zeta_1$$

$$= \int_0^{\zeta} \left(\sum_{p=0} \frac{a_p}{\Gamma(p+1)} \zeta_1^p \right) \left(\sum_{q=0} \frac{b_q}{\Gamma(q+1)} (\zeta - \zeta_1)^q \right) d\zeta_1$$

$$= \int_0^{\zeta} \hat{\varphi}_1(\zeta_1) \hat{\varphi}_2(\zeta - \zeta_1) \, d\zeta_1 = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta)$$ the convolution of $\hat{\varphi}_1$ & $\hat{\varphi}_2$

The Borel transformed model is often refered to as the "Convolutive model"

- Standard observables in QFT when computed in Purturbative regimes take the form $\sum_{n=0}^{\infty} c_n \tilde{z}^{n-1}$ $1/z = g$ when we subtract the tree level resuet.
- Since we know the # of diagrams grows as $n!$ at order n the coefficients are $O(c_n) = O(C^m n!)$
  Hence The Purturbative expansion will only be an asymptotic expansion with zero radius of convergence.

Def$^{\underline{n}}$ 2) The Power series $\tilde{\varphi}(z)$ is of ~~order~~ Gevrey order $1/m$ if the large order terms ~~cn the~~ are bounded by

$$|c_n| \leq \alpha C^n (n!)^m, \quad \alpha, C \text{ Constants.}$$

We will assume that $\tilde{\phi}$ is of Gevrey type 1

$\therefore$ its Borel transform $\hat{\phi}$ defines a convergent expansion at the origin.

In math-land it defines a <u>Germ</u> of analytic functions at $\zeta = 0$

A Germ of analytic functions at z_0 is the set of all the analytic functions with the same taylor expansion around the point z_0.

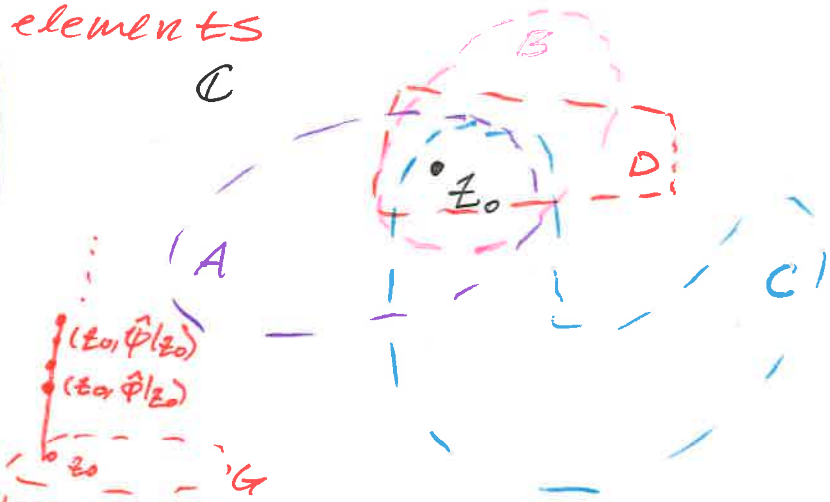
The Germ of analytic functions around the origin will be denoted as $\hat{\phi} \in \mathbb{C}\{\zeta\}$ (or $\hat{\phi}|_{z_0}$ for points not at the origin)

Quick aside: each germ is (named for the germ in a grain of rice or embryo .... it is the early stage or basic genetic information) essentially the "genetic" information of a f^n.

That being said each germ is the set of all analytic f^n's with the same taylor expansion around the point z_0.

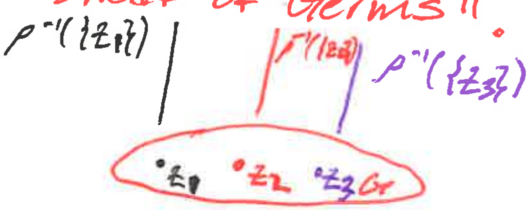
it can be thought of as function elements

$$\hat{\phi}|_{z_0} \simeq \boxed{\begin{array}{c} \cdots(\phi_1, A)\,(\phi_2, B)\cdots \\ \cdots(\phi_3, C)\,(\phi_4, D)\cdots \end{array}}$$



A Stalk for a Given $z_0 \in G \subset \mathbb{C}$ is an ordered pair $(z_0, \hat{\phi}|_{z_0})$ so we can visualize this as "fiber" at z_0

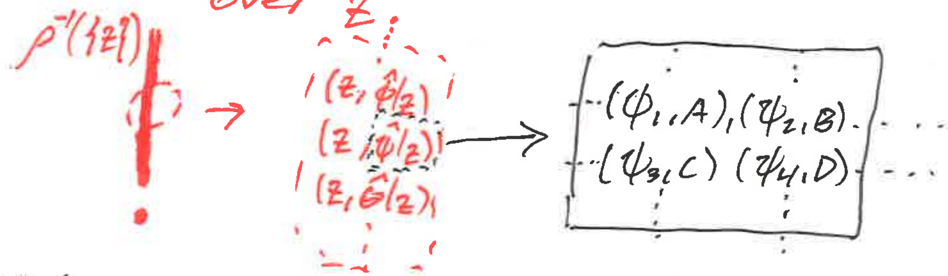
Finally as z varies over G we have a "sheaf of Germs" or a <u>Sheaf</u> of stalks over G



More Precisely: for an open set $G \subset \mathbb{C}$

let $\mathcal{S} = \{(z, \hat{\phi}|_z) \mid z \in G, \hat{\phi} \text{ is analytic at } z\}$

Define $\rho : \mathcal{S}(G) \to \mathbb{C}$ by $\rho((z, \hat{\phi}|_z)) = z$. the Pair $(\mathcal{S}(G), \rho)$ is the Sheaf of Germs of analytic f$\underline{n}$'s on G. $\rho^{-1}(z) = \rho^{-1}(\{z\})$ is the stalk over z.



Back to the Point.

Now that we have improved the convergence of $\tilde{\Phi} \to \mathcal{B}(\tilde{\Phi})$ we need an operator which will bring us back to an analytic extension of The original series.

Def$\underline{n}$ 3.) Define The directional laplace transform

$$\mathcal{L}^{\theta}[\hat{\Phi}](z) = \int_0^{e^{i\theta}\infty} d\zeta\, e^{-z\zeta}\, \hat{\phi}(e^{i\theta}\zeta)$$

$\mathcal{L}^{\theta}$ is linear ; $\mathcal{L}^{\theta}$ maps analytic f$\underline{n}$'s on $e^{i\theta}\mathbb{R}_+$ (with growth rate at most $e^{r|\zeta|}$) into analytic functions $\mathcal{L}^{\theta}\hat{\Phi}$ in The half Plane $\mathrm{Re}(ze^{i\theta}) > r$.

in Particular we can easily compute $\mathcal{L}$ for $\theta=0$ for all ζ^α (monomials)

$$\mathcal{L}^0[\zeta^\alpha] = \int_0^\infty e^{-z\zeta}\zeta^\alpha d\zeta = (-1)^\alpha \int_0^\infty \frac{d^\alpha}{dz^\alpha} e^{-z\zeta} d\zeta$$
$$= (-1)^\alpha \frac{d^\alpha}{dz^\alpha}\left(-\frac{1}{z} e^{-z\zeta}\Big|_0^\infty\right)$$
$$= (-1)^\alpha \frac{d^\alpha}{dz^\alpha}\frac{1}{z} = (-1)^{2\alpha}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{z^{\alpha+1}}$$

This is Precisely The inverse laplace transform

Example) Euler studied

$$\tilde{\Phi}(z) = \sum_{n=0}^\infty (-1)^n n!\, z^{-n-1} \qquad \text{for } z=1$$

$\tilde{\Phi}$ formal solves $\quad \phi'(z) - \phi(z) = -\frac{1}{z} \Rightarrow$

$$\sum_{n=0}^\infty (-1)^n n!(-n-1) z^{-n-2} - \sum_{n=0}^\infty (-1)^n n!\, z^{-n-1}$$
$$= \sum_{n=0}^\infty (-1)^n n!(-n-1) z^{-n-2} - \sum_{n=1}^\infty (-1)^n n!\, z^{-n-1} - \frac{1}{z}$$
$$= -\sum_{n=0}^\infty (-1)^n n!(n+1) z^{-n-2} + \sum_{n=0}^\infty (-1)^n (n+1) n!\, z^{-n-2} - \frac{1}{z}$$
$$= -\frac{1}{z} \checkmark$$

Compute its Borel trans.

$$\hat{\Phi}(\zeta) = \sum_{n=0}^\infty (-1)^n \zeta^n = \frac{1}{1+\zeta}$$

now consider The laplace trans. along $\theta=0$

$$\mathcal{L}(\hat{\Phi}) = \int_0^\infty e^{-z\zeta}\left(\frac{1}{1+\zeta}\right) d\zeta \qquad \zeta = \zeta'-1$$

$$\int_1^\infty e^{z} e^{-\zeta' z}\frac{1}{\zeta'} d\zeta' = e^{z}\int_1^\infty \frac{e^{-z\zeta'}}{\zeta'} d\zeta' \qquad \text{Put } z\zeta' = \xi \qquad d\zeta' = d\xi/z$$
$$= e^{z}\int_z^\infty e^{-\xi}\xi^{-1} d\xi = e^{z}\Gamma(0;z) \leftarrow \text{incomplete gamma fn}$$

expanding around $z\to\infty$ Gives back $\sum_{n=0}^\infty (-1)^n n!\, z^{-n-1}$

$e^{z}\Gamma(0,z)$ is the analytic continuation of $\tilde{\Phi}$

$e^{z}\Gamma(0,z)$ is the particular solution to the equation ~~[illegible]~~ $\phi' - \phi = -\frac{1}{z}$

The ~~ma~~ Homogeneous solution: ce^{z}

Full Solution $\varphi(z) = e^{z}\Gamma(0,z) + ce^{z}$, The Homogeneous solution is not analytic as $z \to \infty$ hence it cannot be captured by our formal series.

Clearly we see that
$$\tilde{\Phi} = \sum_{n=0}^{\infty} C_n z^{-n-1} = \sum_{n=0}^{\infty} C_n \frac{\mathcal{L}^{0}[\zeta^{n}]}{\Gamma(n+1)}$$
$$= \sum_{n=0}^{\infty} C_n \int_0^{\infty} \frac{e^{-z\zeta}\zeta^{n}\,d\zeta}{\Gamma(n+1)}$$
$$= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{C_n}{\Gamma(n+1)} \zeta^{n} \right) e^{-z\zeta}\,d\zeta$$
$$\boxed{\tilde{\Phi} = \mathcal{L}^{0}[\mathcal{B}[\tilde{\Phi}(z)]]}$$

This defines for us a regularization procedure for divergent series.

Closing Statements:

- We computed exactly the Laplace integral for the Borel transform of the formal power series solution ϕ of Eulers eq.
- Generically $\hat{\Phi}$ will not be an analytic f^n along the contour of integration.
- The Resurgent f^n's are a particular class of formal series for which singularities in the ζ-plane (Borel plane) which satisfy certian conditions.
- The behavior of Resurgent f^n close to their singular points will constrain (by using Alien Calculus) the entire structure on the whole Borel plane.

Formal Power series

$$\tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$$

Borel Transformation

Germ of analytic functions at the origin

$$\mathcal{B}[\tilde{\Phi}](\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}$$

Asymptotic Expansion $z \to \infty$

Laplace transform

Analytic function in Region $RE(z) > 0$

$$\mathcal{L}^0[\mathcal{B}[\tilde{\Phi}]](z) = \int_0^{\infty} d\zeta\, e^{-z\zeta}\, \mathcal{B}[\tilde{\Phi}](\zeta)$$