Some extra notes on reduced entropy

History: 1970's- Bekenstein-Hawking black hole entropy
Now understood that in vacuum of QFT
even in flat space a region of space is
mixed state.
Poincaré space in Gibbs state WRT boost gen.

1980's-90's - 't Hooft, Bombelli, Koul, Lee, Sorkin
Susskind, Srednicki

Leading term in entropy of region
is proportional to area. (UV divergent)

Hence entanglement of quantum fields
across the horizon partly explains
BH entropy

90's much work is done on EE in field theory

2000's EE became a standard tool for

CM theorists to characterize phases
of many body systems.

ex: Calabrese-Cardy EE in 2d CFTS

Hastings et al. gapped systems satisfy
an area law.

Kitaev-Preskill/Levin-Wen's topological EE

Li-Haldane's use of entanglement spectrum
to characterize fractional@@ Hall states.
in this decade Ryu-Takayanagi
conjectured a simple formula for EE in
Holographic Theories.

Shannon entropy

Consider a classical theory with a
discrete state space.

Our knowledge of the system, in particular of the state, is described by a probability distribution \( \mathbf{\mathcal{P}} \) with,

\[
\mathcal{P}_a > 0 \quad \sum_a \mathcal{P}_a = 1
\]

The expectation value of an observable is

\[
\langle \mathcal{O}_a \rangle_{\mathcal{P}} = \sum_a \mathcal{O}_a \mathcal{P}_a
\]

The Shannon entropy is

\[
S(\mathcal{P}) = -\sum_a \mathcal{P}_a \ln \mathcal{P}_a
\]

Notice the connection to the Boltzmann entropy.

Supposing all states are equally likely then given the number of states \( \Omega \)

\[
\mathcal{P}_a = \frac{1}{\Omega} \implies -\sum_{a=1}^{\Omega} \frac{1}{\Omega} \ln \frac{1}{\Omega} = \frac{\Omega}{\Omega} \frac{1}{\Omega} (\ln 1 - \ln \Omega) = \frac{\ln \Omega}{\Omega}
\]

The Shannon entropy detects uncertainty in the state:

\[
S(\mathcal{P}_a) = 0 \iff \mathcal{P}_a = \frac{1}{\Omega} \text{ for some state } a
\]

This is true if and only if all observables have vanishing variance \( \Delta \mathcal{O} = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2 = 0 \)

Otherwise \( S(\mathcal{P}) > 0 \)
\[
\langle \mathcal{O}_a^2 \rangle - \langle \mathcal{O}_a^2 \rangle = \sum_{a,b} \mathcal{O}_a \mathcal{P}_a \mathcal{O}_a \mathcal{P}_b - \sum_a \mathcal{O}_a^2 \mathcal{P}_a = \mathcal{P}_a = \mathcal{P}_{aa} = 0
\]

\[
\text{and } S = - \sum \mathcal{P}_a \ln \mathcal{P}_a = - \sum \mathcal{P}_{aa} \ln \mathcal{P}_{aa} = \begin{cases} 
0 & \text{if } a \neq a_0 \\
-\mathcal{P}_{aa_0} \ln \mathcal{P}_{aa_0} & \text{if } a = a_0
\end{cases}
\]

\[
\Rightarrow S = 0 \text{ for } \mathcal{P}_a = \mathcal{P}_{aa_0}
\]

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**Note 2 Things So**

1. **Shannon entropy is extensive**
   i.e. if \( A \uparrow B \) are independent

2. The joint distribution satisfies

\[
\mathcal{P}_{AB} = \mathcal{P}_A \odot \mathcal{P}_B \Rightarrow (\mathcal{P}_{AB})_{ab} = (\mathcal{P}_A)_a (\mathcal{P}_B)_b
\]

Then the entropies add

\[
\mathcal{S}(\mathcal{S}(\mathcal{P}_{AB})) = -\sum_{ab} \mathcal{P}_{AB_{ab}} \ln \mathcal{P}_{AB_{ab}} = -\sum_{ab} \mathcal{P}_a \mathcal{P}_b \ln \mathcal{P}_a + \mathcal{P}_a \mathcal{P}_b \ln \mathcal{P}_b \text{ use } \sum_a \mathcal{P}_a = 1
\]

\[
= -\left(\sum \mathcal{P}_a \ln \mathcal{P}_a + \sum \mathcal{P}_b \ln \mathcal{P}_b\right)
\]

\[
\mathcal{S}(\mathcal{P}_{AB}) = \mathcal{S}(\mathcal{P}_A) + \mathcal{S}(\mathcal{P}_B)
\]

**Note if we have \( N \) independent copies of \( A \) with identical distributions then \( \mathcal{S}(\mathcal{P}_{tot}) = N \mathcal{S}(\mathcal{P}_A) \)**

2. The Shannon Noiseless Coding Theorem states the state of our system can be specified using a binary code (requiring on avg. \( \mathcal{S}(\mathcal{P}_A) \ln 2 \) bits.)

"All forms of data..."
Joint distributions:

Consider a general joint distribution $P_{AB}$

The marginal distribution is defined by integrating out or tracing over 1 of the subsystems

$$(PA)_a = \sum_b (P_{AB})_{ab}$$

$P_a$ gives the right distribution, distribution for any observable which depends only on the set of states $\{a\}$

$$\langle O_{ab} \rangle_{Pab} = \langle O_a \rangle_{Pa}$$

$$\sum_{ab} O_{ab} P_{ab} = \sum_a O_a P_a$$

$$\sum_a \langle O_a \rangle_{Pa} = \sum_a O_a P_a = \langle O_a \rangle_{Pa}$$

For a state $b$ of $B$ with $P_b \neq 0$

The conditional probability on $A$ $P_{A|b}$ is

$$(PA|b)_a = \frac{(P_{AB})_{ab}}{(PB)_b}$$
its entropy avg. over b is

$$\langle S(P_{A|B}) \rangle_p = \sum_b P_b S(P_{A|B})$$

$$= \sum_b P_b (- P_{A|B} \ln P_{A|B})$$

$$= - \sum_{a,b} P_{A|B} \ln P_{A|B} = P_{A|B} \ln P_{A|B}$$

$$= S(P_{A|B}) - S(P_{B})$$

It's easier from this point on to write $S(B) = S(P_{B})$.

This quantity is called the conditional entropy. This is the amount, on average, of entropy which remains to be known about state $a$ after knowing state $B$.

Note that since $H(A|B) = \langle S(P_{A|B}) \rangle_p > 0$

So is $S(AB) - S(B) > 0$ and if $S(AB) = 0$ then so is $S(B)$. This fails in the quantum setting.

A simple example to better understand. This is a phone call between Alice and Bob. When the reception is bad, if the message is $A$, a set of letters and the received message is $B$ another set of letters. Then our machinery helps us to understand the amount of information gained during the call.
The marginal distribution \( P_B \) is the probability distribution describing the probability Bob heard \( B \) when Alice said \( A \).

\[
P_B = \sum_A P_{AB}B
\]

Bob's estimate of the probability that Alice said \( A \) after hearing \( B \) is the conditional probability,

\[
P_{AB} = \frac{P_{AB}B}{P_B}
\]

The Shannon entropy gives, from Bob's point of view, an estimate of the remaining entropy in Alice's signal.

\[
S_{xy} = -\sum_{AB} P_{AB} \ln P_{AB}
\]

and as described on the previous page.

The average over \( B \) of \( S_{xy} \) is the average remaining entropy in Alice's message.

Since \( S_A \) is the total information content about state \( A \), (or Alice's message) \( I \),

\[
S_{AB} - S_B \text{ is the information information Bob still does not know about The message or state A. Then The remaining information Which bob does gain after hearing/observing B is}
\]

\[
I(AB) = S_A - S_{AB} + S_B
\]

This is called the mutual information. It tells us how much we learn about \( A \) by measuring \( B \).
• Why study this problem?

- Intrinsic entropy of a black hole is $S_{BH} = \frac{1}{4} M_{PL}^2 A$
  $M_{PL}$ is the Planck mass, $A$ is the surface area.

- At this time people were still wondering if $S_{BH}$ has anything to do with the #
  of quantum states accessible to the black hole.

- As a black hole shrinks it emits Hawking radiation whose entropy $S_{HR}$ is $S_{HR} = \#S_{BH}$
  where $\#$ is O(1).

- Calculating $S_{HR}$ is done via counting quantum states.

- Obtaining $S_{BH}$ shows getting the amount of missing information represented by $S_{BH}$
  as an answer is what we would expect in a flat space if we did not permit ourselves access to
  the interior of a sphere.
Could talk about a lot of different topics
- Replicatricks, CFT's with holographic duals etc.
- Will be more instructive to focus on a single result.

I constantly read references to early works on entropy in QFT. Decided to discuss
Srednicki 9303048.

Goal: Show that ground state density matrix for a free massless
free field traced over $d\omega_0$+ residing
in a sphere: resulting entropy is
Proportional to area.

Free massless scalar Q.F.
- Represent acoustic modes of a crystal
- Any 3D system with $\omega = c k^2$
- In non-degenerate vacuum state
- Form ground state density matrix $\rho_0 = \langle 0 | 0 \rangle$
- Tr over $d\omega_0$+ inside sphere of radius $R$ $s^2$
- $P_{out}$ resulting density matrix depends on $d\omega_0$+ outside $s^2$
  $S = -\text{Tr} P_{out} \text{S} (R) \propto R^3? R^2$?
Entropy is extensive i.e. depends on system size, i.e. we expect $S \propto N^2$.

We will see that in fact $S = k M^2 A$ with $M$ area $M=\text{UV Cutoff} \propto \text{dimensional constant}$.

Let's start with a simple ex.

two coupled HO (harmonic oscillators)

$$H = \frac{1}{2}(p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2)$$

**Ground State Wave Function**

$$\psi_0(x_1, x_2) = (w + w^-)^{1/4} e^{-\frac{(w + x_1^2 + w - x_2^2)}{2}}$$

$$x_1 = \frac{(x_1 + x_2)}{\sqrt{2}} \quad w_1 = k_0 \frac{x^2}{2} \quad w_2 = (k_0 + 2k_1)^{1/2}$$

**Transformation Matrix**

$$\begin{pmatrix} \delta_{11} \\ \delta_{21} \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_{11} \\ \delta_{21} \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} \delta_{1} + \delta_{2} \\ \delta_{1} - \delta_{2} \end{pmatrix}$$

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = (-i\hbar \partial_1)^2 \begin{pmatrix} \delta_{1} \\ \delta_{2} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = (-i\hbar \partial_2)^2$$

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = -\delta_1^2 = \frac{1}{2} (\delta_{1} + \delta_{2})^2 = -\frac{1}{2} (\delta_{1}^2 + 2\delta_{1}\delta_{2} + \delta_{2}^2)$$

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = -\frac{1}{2} (\delta_{1}^2 - 2\delta_{1}\delta_{2} + \delta_{2}^2)$$

$$H = \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} \delta_1^2 - \frac{i}{\sqrt{2}} \delta_2^2 + k_0(x_1^2 + x_2^2) + 2k_1(x_1 - x_2)^2$$

I will leave proof this is the ground state to you!
Now we can form the density matrix \( \rho = \psi \psi^* \) and trace out the "inside" oscillator leaving the "outside" oscillator.

\[
\rho_{\text{out}} = \int_{-\infty}^{\infty} dx_1 \psi(x_1, x_2) \psi^*(x_1, x_2) \]

\[
= \int_{-\infty}^{\infty} dx_1 \left( \frac{(w+w_+)}{\pi} \right)^{1/2} e^{-\left( \frac{w_+}{2} \left( x_1^2 + z x_1 x_2 + x_2^2 \right) + \frac{w_+}{2} \left( x_1^2 - 2 x_1 x_2 + x_2^2 \right) \right)}
\]

\[
= \left( \frac{(w+w_+)}{\pi} \right)^{1/2} e^{-\frac{1}{4} \left( (w+w_+) x_2^2 + (w_+ + w_-) x_2^2 \right)}
\]

\[
\rho_{\text{out}} = \frac{(2w+w_-)^{1/2}}{(w+w_+)^{1/2}} \left( \frac{\beta}{1 + \beta} \right)^{1/2} e^{-\frac{1}{4} \left( (w_+ + w_-) x_2^2 + (w_+ + w_-) x_2^2 \right)}
\]

\[
\beta = \frac{1}{4} \frac{(w_+ - w_-)^2}{w_+ + w_-}
\]
$$\text{look at } e^{\frac{1}{4}(w^+w^-)X_2^2 + \frac{\beta}{2}X_2^2 + X_2 \rightarrow X'_2 + \beta X_2 X'_2}$$

$$= -\left(\frac{1}{2}(w^+w^-) - \beta\right)X_2^2$$

$$= \left[\frac{\frac{1}{2}(w^+w^-)^2 - 4(w^+w^-)^2}{w^+w^-}\right] X_2^2 + \ldots$$

$$= \frac{1}{2}w^+w^- w^+w^- - \frac{1}{4}w^2 + \frac{1}{2}w^+w^- - \frac{1}{4}w^2$$

$$= w^+w^- + \frac{1}{2}w^+w^- + \frac{1}{4}w^2 + \frac{1}{4}w^2 + w^+w^- - w^+w^-$$

$$= 2w^+w^- - \frac{1}{2}w^+w^- + \frac{1}{4}w^2 + \frac{1}{4}w^2$$

$$= \left[-\gamma \frac{2w^+w^-}{w^+w^-} + \frac{1}{4} \frac{(w^+w^-)^2}{w^+w^-}\right] X_2^2 + X_2 \rightarrow X'_2 + \beta X_2 X'_2$$

$$\gamma - \beta = \frac{2w^+w^-}{w^+w^-} \quad \text{I'll leave this to you to show.}$$

$$\Rightarrow \text{ The full result is}$$

$$P_{\text{out}} = \pi^{-\frac{1}{2}}(\gamma - \beta)^{\frac{1}{2}} e^{-\gamma(X_2^2 + X'_2^2) + \beta X_2 X'_2}$$

Now we would like to find the eigenvalues of $P_{\text{out}}$ (P$n$)
This can be written as an integral eqn.

\[ \int_{-\infty}^{\infty} dx_2 \, \text{Out}(x_2, x_2') f(x_2') = P_n f_n(x_2) \]

i.e. \( \sum_{j} A_{ij} \frac{f_j}{f_i} = P_n f_i \), here we have a continuous label \( i \) instead of integration not summation. If we find them the Von-Neumann entropy is \( S = -\sum P_n \log P_n \).

Consider \( f = 1 \)

\[ \int_{-\infty}^{\infty} dx_2 \, \pi^{-\frac{1}{2}} (y - \beta)^{-\frac{1}{2}} e^{-\gamma(x_2^2 + x_2'^2)} \]

This can be found by guessing (Srednicki is a good guesser!) I will verify the first one and leave the general proof to you. (One way would be induction.)

or you could just do it directly.
Consider \( f_c = e^{-\alpha x^2/2} \), \( \alpha = (\beta - \beta')/2 \)

\[
\int_{-\infty}^{\infty} dx \pi^{-\frac{1}{2}} (\beta - \beta')^{\frac{1}{2}} e^{-\frac{1}{2} \left( x^2 + x'^2 \right) + \beta xx' - \alpha x'^2} \]

Complete the square

\[
- x'^2 \left( \frac{\beta}{\alpha} \right) + \beta xx' - \frac{\beta^2 x^2}{2} \left( \frac{\beta}{\alpha} \right) \frac{1}{2} \]

\[
- \left( \frac{\beta}{\alpha} \right) \left( x'^2 - \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \left( \frac{\beta}{\alpha} \right) \frac{1}{2} \right) \]

\[
\left( \frac{\beta}{\alpha} \right) \left( \left( x' - b \right)^2 - b^2 + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \right) \]

\[
\frac{-1}{2} \left( \frac{\beta}{\alpha} \right) \left( \left( x' - b \right)^2 - b^2 + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \right) \]

\[
\frac{-1}{2} \left( \frac{\beta}{\alpha} \right) \left( \left( x' - b \right)^2 - b^2 + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \right) \]

\[
\pi^{-\frac{1}{2}} (\beta - \beta')^{\frac{1}{2}} \left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{2}} \frac{e + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2}}{2} \]

\[
\left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \alpha x^2} \quad \left( \beta - \beta' \right)^{\frac{1}{2}} \left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} e + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \]

\[
= \left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \alpha x^2} \quad \left( \beta - \beta' \right)^{\frac{1}{2}} \left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} e + \frac{\beta^2 x^2}{\left( \frac{\beta}{\alpha} \right)^2} \]

\[
y + \alpha = \omega^2 + \beta \omega w + \omega^2 + \omega^2 - \omega + \omega^2 \]

\[
y - \beta = \frac{2 \omega w + \omega \omega}{\omega + \omega} \quad \text{Which can be rewritten as}
\]
\[ 1 - \frac{B}{\beta + \alpha} = 1 - \varepsilon \]

altogether \[ \rightarrow \int = (1 - \varepsilon) e^{-\frac{\alpha}{2} x^2} \]

with \( P_0 = (1 - \varepsilon) \)

The full form is \( H_n(x^2) e^{-\frac{\alpha}{2} x^2} = f_n(x) \) eigenfctn's\n
\( (1 - \varepsilon) \varepsilon^n \) eigenvalues.

This shows that \( P_0 \) is equivalent to a thermal density matrix for a single harmonic oscillator with frequency \( \omega \) and temp \( T = \frac{\hbar}{\hbar \omega} \)

\[ \rightarrow \quad P_n = (1 - e^{-\frac{\hbar \omega}{kT}}) e^{-\frac{n \hbar \omega}{kT}} \]

\[ S = - \sum_n P_n \log P_n = - \sum_n (1 - \varepsilon) \varepsilon^n \log (1 - \varepsilon) \varepsilon^n \]

\[ = - \sum_n (1 - \varepsilon) \varepsilon^n (\log \varepsilon^n + \log (1 - \varepsilon)) \]

\[ = (1 - \varepsilon) \log \varepsilon \sum_n \varepsilon^n - (1 - \varepsilon) \log (1 - \varepsilon) \sum \varepsilon^n \]

\[ = (1 - \varepsilon) \log \varepsilon - \frac{\varepsilon \log \varepsilon}{1 + \varepsilon} - \log (1 - \varepsilon) \]

\[ S \text{ is only a ratio of } K_i / K_0 \]
Now extend our analysis to $N$ coupled harmonic oscillators.

\[ H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i,j} x_i x_j \kappa_{ij} \]

$\kappa$ is a real symmetric matrix with positive eigenvalues.

For $2$ harmonic oscillators our ground state wave function was.

\[ \Psi_0 = \pi^{-\frac{1}{2}} (\omega^+ + \omega^-)^{\frac{1}{4}} e^{-\frac{1}{2} (\omega^+ x^2 + \omega^- x^2)} \]

For $N$ harmonic oscillators we can see a simple generalization $\omega^+ = \kappa \omega^\frac{1}{2}$ $\omega^- = (\kappa_0 + 2 \kappa_1) \omega^\frac{1}{2}$

\[ \kappa_{ij} = (\kappa_0 + \kappa_1 - \kappa) \\
\Rightarrow \det \kappa_{ij} = \kappa_0^2 + 2 \kappa_0 \kappa_1 + \kappa_1^2 - \kappa^2 = \kappa_0 (\kappa_0 + 2 \kappa_1) \]

\[ (\omega^+ + \omega^-)^{\frac{1}{4}} = (\sqrt{\det \kappa_{ij}})^{\frac{1}{4}} \]

If $\kappa_{ij} = \sin \kappa_{ij}$ with $\kappa_0 \geq \kappa_1$ is the "square root of $\kappa$".

Then \((\omega^+ + \omega^-)^{\frac{1}{4}} = (\det \omega)^{\frac{1}{4}} \]

Note that if $\kappa = U^T \kappa_0 U$ with $\kappa_0$ diagonal,

\[ \omega^+ = U^T \kappa_0^\frac{1}{2} U \Rightarrow \kappa = U^T \kappa_0^\frac{1}{2} U U^T \kappa_0^\frac{1}{2} U \]

Note also

\[ \omega^+ x_i^2 + \omega^- x_i^2 = \omega_{ij} x_i x_j \]

with $x_i = U_{ij} x_j$.

\[ \Psi_0 = \pi^{-\frac{1}{2}} (\det \omega)^{\frac{1}{4}} e^{-\frac{1}{2} (x_i \omega_{ij} x_j)} \]

$x_i = (x_1, x_2)$

$x_i = (x_1, x_2)$
Now we follow the same steps. Trace out the first n "inside" oscillators

\[ P_{out} (x_{n+1}, \ldots, x_N, x'_{n+1}, \ldots, x'_N) = \prod_{i=1}^{n} x_i \psi_0 (x_i, \ldots, x_n, x'_{n+1}, \ldots, x'_N) \]

The full ground state wave \( \psi_0 = \prod_{i=1}^{N} e^{-\frac{\nu}{2} (x_i \cdot \Omega^{-1} \cdot x_i')} \) on the back page.

To solve these integrals we can motivate ourselves with the previous solution

\[ e^{-\frac{\nu}{2} (x_0^2 - x_0'^2)} \]

Notice \( \beta = \frac{1}{4} \frac{(w_+ - w_-)^2}{w_+ + w_-} \)

\[ \begin{align*}
\gamma & = \frac{2w_+ w_-}{w_+ + w_-} \\
\Theta & = \begin{pmatrix}
w_+ + w_- \\
w_+ - w_- \\
A (n \times n) \\
B (n \times n-n) \\
C (n-n \times n-n)
\end{pmatrix}
\end{align*} \]

\[ \beta = \frac{1}{4} (w_+ - w_-) \begin{pmatrix} w_+ + w_- \end{pmatrix} \]

\[ \beta^2 = \frac{1}{4} B^T A^{-1} B \]

So if you look back to the calculation of \( \psi_{out} \) for 2 oscillators,

\[ \beta = \frac{2w_+ w_-}{w_+ + w_-} \]

You can kinda see this.
We don't need to worry about the normalization since eigenvalues (\(\mu_i\)) \(\mu_n \to \Delta \mu_n = 1\)

Generalizing
\[
\lim_{n \to \infty} \prod_{i=1}^{n} \phi_n(x_i - x_{n-i}, x'_i - x'_{n-i}) \phi_n(x_i, x'_i)
\]

\[= \mu_n \phi(x_i - x_{n-i})
\]

We can see that \(\det G / \phi_n(G x, G x')\) has the same eigenvalues.

Let \(y = V^T y_0 \downarrow V\), \(V^T V = I\), \(y_0\) - diagonal

and put \(x = V^T y_0^{-\frac{1}{2}} y\)

\[\Rightarrow e^{-\frac{1}{2} x \cdot y \cdot x + x \cdot y' \cdot x'} + \phi \cdot x\]

\[= e^{-y \cdot x \cdot y_0^{-\frac{1}{2}} y + x \cdot y_0^{-\frac{1}{2}} y' \cdot y_0^{-\frac{1}{2}} y'} + \phi \cdot y_0^{-\frac{1}{2}} y \cdot y_0^{-\frac{1}{2}} y' \cdot y_0^{-\frac{1}{2}} y'
\]

\[= e^{-\frac{1}{2} y_0^{-\frac{1}{2}} y \cdot y_0^{-\frac{1}{2}} y' \cdot y_0^{-\frac{1}{2}} y' \cdot y_0^{-\frac{1}{2}} y'} + \phi \cdot y_0^{-\frac{1}{2}} y \cdot y_0^{-\frac{1}{2}} y' \cdot y_0^{-\frac{1}{2}} y'
\]

Now choose \(y = w^2\) s.t. \(w^T \phi' w = \phi_0\)

\[\Rightarrow e^{-\frac{1}{2} x \cdot z + x \cdot z' - \frac{1}{2} z^2 + x \cdot z' + 2 \cdot \beta_0 \cdot z'}
\]

\[= \prod_{i=n+1}^{N} e^{-\frac{1}{2} (z_i + z'_i)^2 + 2 i z_i \beta_i} \Rightarrow \beta_i \rightarrow \beta'_i \] (i.e., \(p'_0 = \prod_{i=1}^{N} \beta_i^\beta_i \beta_0^\beta_0\))

This is exactly as we found before with \(\delta \rightarrow 1\) \(\Rightarrow \beta \rightarrow \beta_i\)

Therefore the entropy is a sum over the entropy of each independent term.
\[ P_n = \prod_{i} (1 - \xi_i)^{\xi_i^n} \quad \Pi = \prod_{i=m+1}^{N} \]

\[ S = - \sum_{n=0}^{\infty} P_n \ln P_n = - \sum_{n} P_n \ln(\Pi_j (1 - \xi_j)^{\xi_j^n}) \]

\[ = - \sum_{n} P_n \left( \sum_{j} \ln(1 - \xi_j) + \ln(\xi_j^n) \right) \]

\[ = - \sum_{n} \left( \Pi_{\xi} (1 - \xi_i)^{\xi_i^n} \sum_{j} \ln(1 - \xi_j) + \Pi_{\xi} (1 - \xi_i)^{\xi_i^n} \sum_{j} \ln(\xi_j^n) \right) \]

\[ = - \sum_{n} \left( \Pi_{\xi} (1 - \xi_i)^{\xi_i^n} \sum_{j} \frac{\ln(1 - \xi_j)}{1 - \xi_i} + \Pi_{\xi} (1 - \xi_i)^{\xi_i^n} \sum_{j} \frac{\ln(\xi_j^n)}{1 - \xi_i} \right) \]

\[ S = - \sum_{j} \left( \ln(1 - \xi_j) + \left( \prod_{i} \frac{\xi_i}{(1 - \xi_i)^{\xi_i^n}} \times \ln(\xi_j^n) \right) \right) \]

\[ \text{Careful, I think I made a mistake in this derivation.} \]
Now we want to apply this to a scalar quantum field.

\[ L = \frac{1}{2} \gamma^\mu \partial_\mu \phi \partial_\nu \phi \]  

use \( \gamma = \text{diag}(1, -1, -1, -1) \)

\[ \pi = \frac{\partial L}{\partial \dot{\phi}} = \partial \dot{\phi} = \dot{\phi} \]

\[ H = \pi \dot{\phi} - L = \pi \dot{\phi} - \frac{1}{2} \gamma^\mu \partial_\mu \phi \partial_\nu \phi \]

\[ H = \pi^2 - \frac{1}{2} \pi^2 + \frac{1}{2} \partial \dot{\phi} \partial \dot{\phi} \]

\[ H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 \]

\[ H = \oint d^3 x \pi \dot{\phi} = \frac{1}{2} \oint d^3 x \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 \]

We will need to regulate the infinities so introduce partial waves.

\[ \Psi_{lm} = x \int d \Omega \, \tilde{\Psi}_{lm}(\theta, \phi) \phi(x) \]

\[ \Pi_{lm} = x \int d \Omega \, \tilde{\Pi}_{lm}(\theta, \phi) \pi(x) \]

\( \tilde{\Psi}_{lm} - \text{real spherical harmonics} \)

\( \tilde{\Pi}_{lm} = \text{Lorentz scalar} \)

\[ \tilde{\Psi}_{lm} = Y_{lm}, \quad \tilde{\Pi}_{lm} = \sqrt{2} \text{Re} \, Y_{lm} \]

\[ \oint d \Omega \, Y_{lm}^* Y_{lm'} = \delta_{ll'} \delta_{mm'} \]
We have \[ \langle \phi(x), \pi(x') \rangle = \int \delta(x - x') \text{ This is 3D.} \]

\[ \langle \phi(x_1), \pi(x_1) \rangle = \int \frac{d^3\Omega}{4\pi} \mathcal{Z}(\Omega, \phi(x_1), x_1) \mathcal{Z}(\Omega', \phi(x_1), x_1) \]

\[ = \int d\Omega d\Omega' \left( \frac{d}{d\Omega} \mathcal{Z}(\Omega, \phi(x_1), x_1) \mathcal{Z}(\Omega', \phi(x_1), x_1) \right) \]

\[ = \delta(x - x_1) \int d\Omega d\Omega' \frac{d}{d\Omega} \mathcal{Z}(\Omega, \phi(x_1), x_1) \mathcal{Z}(\Omega', \phi(x_1), x_1) \]

\[ = \delta(x - x_1) \int d\Omega \mathcal{Z}(\Omega, \phi(x_1), x_1) \]

\[ \phi_\ell^m(x) = \int d\Omega \mathcal{Z}(\Omega, \phi(x_1), x_1) \]

\[ \phi_\ell^m(x) \rightarrow \phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell^m}(x) \mathcal{Z}(\ell, m) \]
Important: as $H = \frac{1}{2} m \dot{\Phi}_{\ell m}$

with $\Phi_{\ell m} = \frac{1}{2} \int_0^\infty dx \left[ \frac{(\dot{\Phi}_{\ell m}(x))^2}{x^2} + x^2 \left( \frac{\partial}{\partial x} \left( \frac{\Phi_{\ell m}(x)}{x} \right) \right)^2 + \frac{(\ell + 1)}{x^2} \Phi_{\ell m}^2 \right]$

we have made no approximations of regulations.

Regulators: ultraviolet = replace radial $x$ by a lattice of discrete points with spacing $a \Rightarrow$ UV cutoff is $\mu = a$.

Infrared = put the system in a spherical box of radius $L = (N+1)a/N$ is a large integer $\Rightarrow \Phi_{\ell m}(x) = 0 \text{ for } x \geq L$

IR cutoff $\mu = L^{-1}$
\[ H_{m} = \frac{1}{2 \lambda} \sum_{j=1}^{N} \left[ \pi_{m,j}^{2} + (1 + \frac{\lambda}{2})^{2} \left( \frac{\phi_{m,j}}{j} - \frac{\phi_{m,j+1}}{j+1} \right) + \frac{\ell \xi_{j+1}}{j} \right] \]

Hamiltonian has general form of
\[ H = \frac{1}{2} \sum_{i=1}^{N} p_{i}^{2} + \frac{1}{2} \sum_{i,j} \lambda_{ij} x_{i} x_{j} x_{j} \]

Can numerically compute \( S_{m} \) at fixed \( N \)
(I tried to reproduce this but ran out of time)

Note: \( S = \sum_{m} S_{m}(n, N) \)
\( H_{m} \) is independent of \( m \)
\[ \Rightarrow S_{m} = S_{m} \sum_{k} (2 \ell + 1) S_{k}(n, N) \]

\( S_{k} \) can be computed perturbatively in the limit \( \ell \gg N \)
\( S_{k}(n, N) \) is independent of \( \ell \) here
\[ S_{k}(n, N) = \xi_{k}(n) [-\log \xi_{k}(n) + 1] \]
\[ \xi_{k} = \frac{n(n+1)(2n+1)^{2}}{64 \ell^{2}(\ell+1)^{2}} + O(\ell^{-6}) \]

→ These can be used to check numerical results.
\[ R = (n + \frac{1}{2})a \], radius midway between outermost point which was traced over \( \frac{1}{2} \) innermost point which was not traced out d.o.f.'s

See Page 8 of attached reference for plot of \( S vs R^2 \) which I did not have time to reproduce.

The author does this for \( N = 60 \) \( \leq n \leq 30 \) fitting the data gives

\[ S = 0.3 M^2 R^2, \quad M = \pm 1 \]

Notes:
The linear behavior cannot continue to \( n = N \), i.e., we have traced out all d.o.f.'s!
\[ S = 0 \quad S \to 0 \text{ as } R \to L = (N + 1)a \]

Summary:
- Counting quantum states in a simple setup
- Produced a reduced (Entanglement) entropy proportional to surface area of inaccessible region

The wall of our Spherical box.