University of Alabama Department of Physics & Astronomy
Graduate Qualifying Examination
Classical Mechanics

9 January 2019

General Instructions

- No reference materials are allowed.
- Do ***all*** your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your ***assigned number*** and the subject.
- Turn in the questions for each part with the answer booklet.
- 150 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.
- Answer 5 out of the 6 problems and clearly indicate the problems you wish to be graded.

1. Three equal masses m are placed in contact with each other forming a straight line segment. Another mass $M(\neq m)$ strikes a mass m at the end of the line segment with a speed v. This collision is one-dimensional and elastic.

   (a) (8 points) Considering the conservations of momentum and energy of the system, show that at least two masses are in motion after the collision. (Hint: show that no solution exists to satisfy the momentum and energy conservations if we assume only one mass is in motion after the collision.)

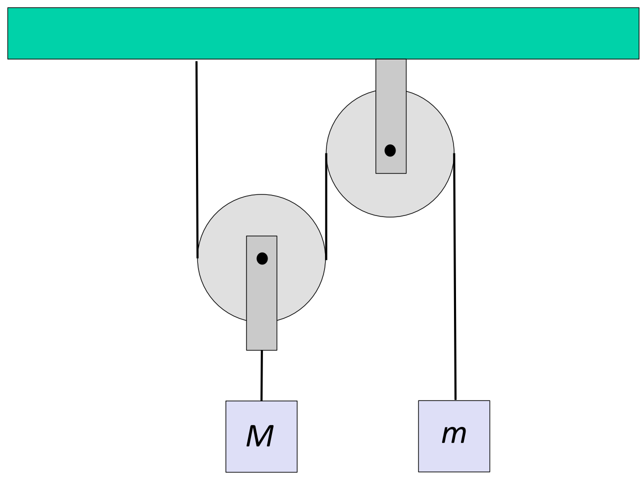
   (b) (6 points) Consider the case $M < m$. If only two masses are in motion after the collisions, which ones are they and what are their speeds?

   (c) (6 points) Solve the same problem as (b), but for the case $m < M < 2m$.

2. In the pulley system shown in the figure, two masses (M and m) are connected with a massless string. We assume that the string does not stretch and the pulleys are massless and frictionless.

   (a) (10 points) Express the acceleration of the mass M in terms of M, m and g (the free-fall acceleration).

   (b) (10 points) Find the tension of the center string.



3. Under a central force, an object of mass m follows a path which in polar coordinates is given by $r(\theta) = r_0\,\theta$, where r_0 is a constant.

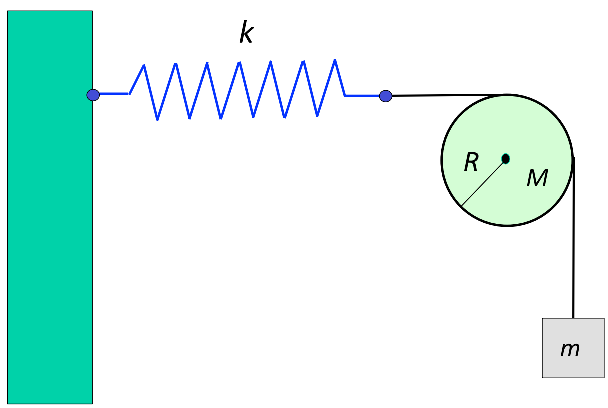
(a) (10 points) In this system, the energy (E) and the angular momentum (L) are conserved. For given E and L, find the potential $V(r)$ leading to such an orbit.

(b) (10 points) Find r and θ as a function of the time along the orbit. Here, set the initial condition as $\theta(t = 0) = 0$.

4. Consider a system that a spring with a force constant k is connected to a mass m by a massless string passing over a frictionless pulley with mass M and radius R (see the figure below). We assume that the string does not stretch and moves without slipping over the pulley.

(a) (10 points) The mass m can oscillate around its equilibrium position in the vertical direction. When the pulley is massless ($M = 0$), find the period of the oscillation in terms of m, k and g (the free-fall acceleration).

(b) (10 points) When $M \neq 0$, find the period of the oscillation in terms of m, M, k and g. In solving the problem, use the moment of inertia of the pulley, $I = MR^2/2$.



5. Consider the motion of two particles in 2 dimensional space (x-y plane). We express the position vectors of the particle 1 and 2 with masses m_1 and m_2, respectively, as

$$\vec{x}_1 = (x_1, y_1) \ \text{ and } \ \vec{x}_1 = (x_2, y_2).$$

Suppose the potential of the system depends only on the distance between the two particles, $V(\ell)$, where $\ell = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

(a) (6 points) Write the Lagrangian of the system.

(b) (6 points) From the Lagrangian, write the Euler-Lagrange equations for the two particles (4 equations for x_1, y_1, x_2 and y_2). Since a concrete formula for the potential V is not given, use $\frac{dV(\ell)}{d\ell}$.

(c) (4 points) Show that the total momentum of the system is conserved.

(d) (4 points) Show that the total angular momentum of the system is conserved.

6. The Lagrangian for a relativistic particle in 1-dimensional space is given by

$$L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}}, \tag{1}$$

where c is the speed of light (constant) and x is the position of the particle.

(a) (8 points) Write down an expression for the particle momenta p, and show from the Euler-Lagrange equation that it is conserved.

(b) (6 points) From the resulting Hamiltonian, derive the expression for the conserved energy of the form

$$E = \frac{mc^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}. \tag{2}$$

(c) (6 points) Use your results for (a) and (b) to verify that $E^2 = m^2c^4 + p^2c^2$.

[label=**0**:]**Energy of a current distribution [20 points]: This problem is concerned with a static current distribution, characterized by a (static) current density $\mathbf{j}(\mathbf{r})$, and the magnetic field $\mathbf{B}(\mathbf{r})$ produced by it. Starting from the expression for the energy stored in the magnetic field $W_{mag} = \frac{1}{2\mu_0}\int d\mathbf{r}\ \mathbf{B}^2(\mathbf{r})$, the aim is to write W_{mag} in terms of the current densities. To this end, work through the following steps:**

1. (a) **Show that in the Coulomb gauge the vector potential $\mathbf{A}(\mathbf{r})$ fulfills the equation $\nabla^2\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{j}(\mathbf{r})$, and find the solution to this equation. [5 points]**

   (b) **Show that the magnetic energy can be written in terms of the currents as $W_{mag} = \frac{\mu_0}{8\pi}\int d\mathbf{r}d\mathbf{r}'\ \frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$. HINTS: It is convenient to express one of the magnetic fields in the original expression for W_{mag} in terms of the vector potential and to make use of the relation derived in (a). [15 points]**

2. Wave equations in vacuum [20 points]: Starting from the Maxwell equations in vacuum, show that the fields E and B fulfill the *inhomogeneous* wave equations

(a) [10 points]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = \lambda_1(\mathbf{r},t) \tag{1}$$

(b) [10 points]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \lambda_2(\mathbf{r},t) \tag{2}$$

and determine λ_1, λ_2 and c.

3. EM wave [20 points]: Consider a linear, uncharged, and isotropic insulator. Suppose that the magnetic field

$$\mathbf{B}(\mathbf{r},t) = \Re[\mathbf{B}_0(\mathbf{r},t)\mathbf{e}^{-i\omega t}], \quad \mathbf{B}_0(\mathbf{r},t) = (\alpha\hat{\mathbf{e}}_x + i\gamma\hat{\mathbf{e}}_y)\mathbf{e}^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3}$$

where α and γ are real constants, fulfills the homogeneous wave equation (and, of course, the Maxwell equations).

(a) State the Maxwell's equations in differential form applicable for this case in terms of E and B, as well as the relation of E and B to D and H. [4 points]

(b) State the wave equation for B. What is the speed u of the wave? What is the relation between ω and k? [4 points]

(c) Find the direction of k. [4 points]

(d) Find the electric field $\mathbf{E}(\mathbf{r},t)$. You may use the ansatz $\mathbf{E}(\mathbf{r},t) = \Re[\mathbf{E}_0(\mathbf{r})\mathbf{e}^{-i\omega t}]$. [4 points]

(e) Find the average energy density $\overline{W}$ of the electromagnetic field expressed through α and γ. [4 points]

4. Electric field of a square sheet [20 points]:

(a) Show that the electric field at a height z above the center of a square sheet (side a) carrying a uniform surface charge σ is given by: [15 points]

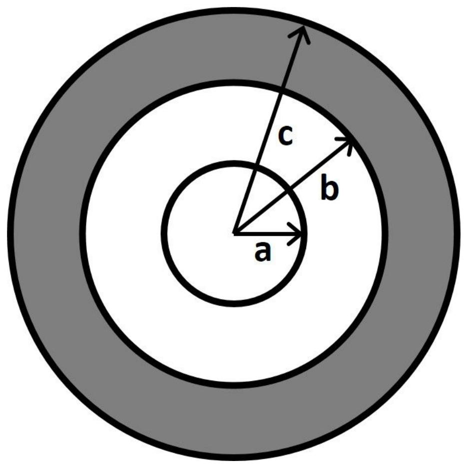
$$\mathbf{E}(z) = \left(\frac{\sigma}{2\epsilon_0}\right)\left[\frac{4}{\pi}\arctan\sqrt{1+\left(\frac{a^2}{2z^2}\right)}-1\right]. \tag{4}$$

(b) Check your result for the limiting case a $\rightarrow \infty$. [5 points]

5. **Capacitance of concentric cylinders [20 points]: A certain coaxial cable consists of a cooper wire, radius a, surrounded by a concentric copper tube of inner radius c (see the figure below). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown.**

   (a) **Find the capacitance per unit of length of this cable. [15 points]**

   (b) **What would the capacitance if the dielectric between the cylinders is removed? [5 points]**



6. Magnetic fields in matter [20 points]: An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis, M=$ks\hat{\mathbf{z}}$, where k is a constant and s is the distance from the axis; there is no free current anywhere. Using the Ampere's law, find the magnetic field:

(a) Inside the cylinder. [10 points]

(b) Outside the cylinder. [10 points]

University of Alabama Department of Physics & Astronomy
Graduate Qualifying Examination
Quantum Mechanics

20 August 2019

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet (no scratch paper is allowed).
- On the cover of each answer booklet put only your ***assigned number*** and the subject.
- Turn in the questions for each part with the answer booklet.
- 150 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.
- Answer 5 out of the 6 problems and clearly indicate the problems you wish to be graded.

1. The wave function for a particle in an infinite square well with boundaries at $x = 0$ and $x = a$ is

$$\psi(x) = Ax(a - x) \; , \qquad 0 \leq x \leq a \; ,$$

and zero elsewhere.

(a) (6 points) Find A.

(b) (6 points) Compute the expectation value and uncertainty in a measurement of position.

(c) (6 points) Compute the expectation value and uncertainty in a measurement of momentum.

(d) (2 points) Verify that the Heisenberg uncertainty relation holds.

2. Consider the operator $\hat{O} = \hat{x}\hat{p} + \hat{p}\hat{x}$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, respectively, for a free particle moving in one dimension.

(a) (10 points) Can $\hat{O}$ be measured simultaneously with the energy E? If not, find a lower bound on the product of their uncertainties, $\Delta O \Delta E$.

(b) (10 points) In the absence of a measurement, show that the expectation value of $\hat{O}$ changes linearly in time, $< \hat{O} > (t) = c_1 + c_2 t$, where c_2 depends on the mean energy.

3. Say that the angular momentum operators for the x and z components of a particle are represented by the matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$

respectively

(a) (7 points) What is the matrix representation for the y component of the angular momentum S_y?

For the following two parts, assume that the system is in a state where there is a 50% probability to measure S_z to be $\hbar$, and a 50% probability to measure S_z to be $-\hbar$.

(b) (6 points) What is the expectation value of S_z, and what is the likelyhood that one can obtain that value in a measurement?

(c) (7 points) What is the expectation value of S_x, and what is the likelyhood that one can obtain that value in a measurement?

4. A particle moves in one-dimension and in the presence of the potential energy

$$V(x) = \kappa(x^2 - \ell^2)^2 \,, \qquad \kappa > 0$$

(a) (10 points) Find the locations of the stable equilibrium(a) of the particle.

(b) (10 points) Say that the particle undergoes *small* oscillations about the stable equilibrium (a). Give an approximate value for the quantum ground state energy, and give a condition on κ for which the small oscillation approximation is valid.

5. In a 2-dimensional system, a particle of mass m is trapped in an infinite potential well,

$$V(x,y) = \begin{cases} 0 & (|x| \leq L,\ |y| \leq 2L) \\ \infty & (|x| > L,\ |y| > 2L) \end{cases} .$$

(a) (10 points) Write the solution of the time-independent Schrödinger equation in the region inside the well.

(b) (10 points) Determine the energy spectrum of the system.

6. Consider the system with two states based on the Hamiltonian $H = H_0 + \epsilon H_1$,

$$H_0 = E_0 \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

where $0 < \epsilon \ll E_0$.

(a) (5 points) We first consider the limit of $\epsilon = 0$. Find two eigenvalues and corresponding eigenstates in the form of column vector.

(b) (5 points) Find the energy eigenvalues at the first order perturbation with respect to $\epsilon / E_0 \ll 1$.

(c) (5 points) In fact, we can easily diagonalize the 2×2 matrix H. Find the exact eigenvalues.

(d) (5 points) Using the result in (c), verify the result in (b) at the 1st order of $\epsilon / E_0 \ll 1$.

University of Alabama Department of Physics & Astronomy Graduate Qualifying Examination Thermal Physics

21 August 2019

General Instructions

Answer 2 out of 3 of the problems. Clearly indicate on the inside cover of the answer booklet the problems you wish to be graded.

- No reference materials are allowed.
- Do all your work in the corresponding answer booklet.
- No scratch paper is allowed.
- On the cover of each answer booklet put only your assigned number and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 90 minutes are alloted for this part.
- No electronic devices are allowed. In particular, handheld computers, PDAs, and cell phones are explicitly prohibited.

1: The Brayton Cycle [50 pts]

The Brayton or Joule cycle consists of two isentropic and two isobaric steps. In a working engine air (and fuel) is compressed adiabatically (A $\rightarrow$ B), heated by fuel combustion at constant pressure (B $\rightarrow$ C), expanded (C $\rightarrow$ D), and rejected to the atmosphere. The process (D $\rightarrow$ A) occurs outside the engine, and a fresh charge of air is taken in to repeat the cycle.

Assume the working gas is a monoatomic ideal gas with the following equations of state:

$$PV = NRT, \qquad U = \alpha NRT, \quad \text{where } \alpha = 3/2.$$

(a) Sketch the Entropy-Pressure diagram for this cycle. [15 pts]

Find the heat added and the work done by the gas in terms of temperature as the engine goes

(b) from A $\rightarrow$ B (Adiabatic compression). [5 pts]

(c) from B $\rightarrow$ C (Isobaric Heating - expansion). [5 pts]

(d) from C $\rightarrow$ D (Adiabatic expansion). [5 pts]

(e) from D $\rightarrow$ A (Isobaric cool-down - compression). [5 pts]

(f) Show that the engine efficiency is given by: [15 pts]

$$\varepsilon = \frac{W_{net}}{Q_{in}} = 1 - \left(\frac{P_A}{P_B}\right)^{\frac{c_P - c_V}{c_P}}$$

Remember that the engine efficiency is defined as (net work)/(total heat in), where the (total heat in) includes only the parts of the cycle in which heat flows into the gas and excludes parts in which heat flows out.

2: Classical gas in a gravitational potential [50 pts]

Consider a gas of non-interacting particles in the gravitational potential $V_{grav}(x, y, z \geq 0) = mgz$, with the gravitational constant $g > 0$ at fixed temperature T. The volume V of the gas is confined to a vertical, cylindrical vessel of radius R and of semi-infinite height.

(a) Using the canonical ensemble, find the Helmholtz free energy, the entropy, and the internal energy of this system. You may use the Stirling approximation $\log N! = N \log N - N$ for $N \gg 1$. [30 pts]

(b) Calculate the heat capacity. [10 pts]

(c) Find the local particle density at height z, $n(z)$, normalized such that $N = \int d^3q n(q_z)$. [10 pts]

3: Ideal fermi gas in a magnetic field [50 pts]

Consider an ideal Fermi gas of N spin 1/2 particles in a volume V and in a magnetic field H, oriented in z-direction, at temperature $T = 0$. The energy of a particle is

$$\varepsilon = \frac{p^2}{2m} - \sigma \mu_B H \tag{1}$$

where μ_B is Bohr's magneton, and $\sigma = \pm 1$ for spin states $\uparrow$ and $\downarrow$ with respect to the z-axis.

(a) Find the chemical potential μ_0 in terms of the number of particles N for vanishing magnetic field. [10 pts]

(b) Write down expressions for the Fermi momenta of the spins oriented parallel (p_{F-}) and antiparallel (p_{F+}) to the external magnetic field $H \neq 0$, dependent on the chemical potential μ at finite magnetic field. [10 pts]

(c) Find the total energy of spins oriented parallel (E_-) and antiparallel (E_+) to the magnetic field $H \neq 0$, in terms of the momenta p_{F+} and p_{F-}. [10 pts]

(d) Show that $\mu \approx \mu_0$ up to terms of order $\mathcal{O}(H^2)$, i.e., that the chemical potential is only weakly affected by the magnetic field. [10 pts]

(e) Find the susceptibility $\chi = \partial M/\partial H|_{H=0}$. Here, M is the average magnetization per volume, $M = \mu_B(N_- - N_+)/V$, and $N_\mp$ are the numbers of fermions with spin pointing parallel and antiparallel to the magnetic field. [10 pts]