Effective Lagrangian of the EM field in QED. and optical birefringence of the vacuum

Recent Connected Works

1) Magnetic Instability in AdS/CFT
2) Vacuum Instability in Electric fields via AdS/CFT.
3) EM Instability and Schwinger effect in Witten-Sakai-Sugimoto model.
4) Extension to the case of an effective lagrangian for supersymmetric QCD.

The Schwinger effect is a predicted physical phenomenon whereby matter is created by a strong electric field. It is a prediction of QED in which $e^- e^+$ pairs are spontaneously created in the presence of an electric field, thereby causing decay of the electric field.

1936 → Heisenberg, Weisskopf. [Physical ideas based on Dirac's hole theory].

1951 → complete theoretical description by Schwinger. [This description is more refined and based on principles of QFT].
Solution of Dirac equation in a Homogeneous Magnetic Field

\[
\begin{align*}
\psi(t) &= (c\alpha \cdot \hat{p} + \beta mc^2) \psi \\
\epsilon(x) &= \sigma \cdot \hat{p}(x) + m_0 c^2 \sigma \cdot (0-1)(\chi) \\
\epsilon \phi &= \sigma \cdot \hat{p} \phi + m_0 c^2 \phi \\
\epsilon \chi &= \sigma \cdot \hat{p} \chi - m_0 c^2 \chi
\end{align*}
\]

Usual splitting up the four component \(\psi\) Dirac bispinor into two two component spinors \(\phi\) and \(\chi\).

Introducing the electromagnetic potentials into Dirac equation,

\[
\begin{align*}
\Psi(t) &= (c\alpha \cdot (\hat{p} - \frac{e}{c} A) + \beta e A_0 + \beta mc^2) \psi \\
\hat{\Pi} &= \hat{p} - \frac{e}{c} A \\
\alpha_1 &= \left( \begin{array}{cc} 0 & \sigma^1 \\ \sigma^1 & 0 \end{array} \right) \\
\epsilon (\hat{\phi}) &= \left( \begin{array}{cc} c & \sigma \cdot \hat{\chi} \\ \sigma \cdot \hat{\phi} & -c \end{array} \right) + eA_0 \left( \begin{array}{cc} \hat{\phi} \\ \hat{\chi} \end{array} \right) + m_0 c^2 \left( \begin{array}{cc} \hat{\phi} \\ \hat{\chi} \end{array} \right) \quad \text{c = 1} \\
\epsilon (\hat{\phi}) &= \left( \begin{array}{cc} c & \sigma \cdot \hat{\chi} \\ \sigma \cdot \hat{\phi} & -c \end{array} \right) + eA_0 \left( \begin{array}{cc} \hat{\phi} \\ \hat{\chi} \end{array} \right) + m_0 c^2 \left( \begin{array}{cc} \hat{\phi} \\ \hat{\chi} \end{array} \right)
\end{align*}
\]

Minimal coupling procedure

\[
\begin{align*}
\frac{\partial \psi}{\partial t} &= (c\alpha \cdot (\hat{p} - \frac{e}{c} A) + \beta e A_0 + \beta mc^2) \psi \\
\text{Set } \hbar &= 1 \\
\text{Kinetic momentum } \hat{\Pi} &= \hat{p} - \frac{e}{c} A \\
\alpha_1 &= \left( \begin{array}{cc} 0 & \sigma^1 \\ \sigma^1 & 0 \end{array} \right) \\
\frac{\partial \hat{\phi}}{\partial t} &= \sigma \cdot \hat{\Pi} \hat{\phi} + eA_0 \hat{\phi} + m_0 c^2 \hat{\phi} \\
\frac{\partial \hat{\chi}}{\partial t} &= \sigma \cdot \hat{\Pi} \hat{\chi} + eA_0 \hat{\chi} - m_0 c^2 \hat{\chi}
\end{align*}
\]

Stationary solutions for a purely magnetic field \((A_0 = 0, A \text{ independent of time})\).

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= \hat{\sigma} \cdot (\hat{p} - eA) \phi + m \phi \\
\frac{\partial \chi}{\partial t} &= \hat{\sigma} \cdot (\hat{p} - eA) \chi + eA_0 \chi - m \chi \\
(\epsilon + m) \phi &= \hat{\sigma} \cdot (\hat{p} - eA) \phi \quad - (1) \\
(\epsilon + m) \chi &= \hat{\sigma} \cdot (\hat{p} - eA) \chi \quad - (2) \\
(\epsilon + m) = \hat{\sigma} \cdot (\hat{p} - eA) \phi
\end{align*}
\]

Multiply \(1\) by \((\epsilon + m)\) and eliminate \(\chi\),

\[
(\epsilon^2 - m^2) \phi = \hat{\sigma} \cdot (\hat{p} - eA) \hat{\sigma} \cdot (\hat{p} - eA) \phi
\]
\[(\varepsilon^2-m^2)\phi = \hat{\sigma} \cdot (\hat{\rho} - eA) \hat{\sigma} \cdot (\hat{\rho} - eA) \phi - 4\]

Identity: \((\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i \sigma \cdot a \times b\)

\[(\varepsilon^2-m^2)\phi = \left[ (\rho - eA)^2 + i \sigma \cdot (\rho - eA) \times (\rho - eA) \right] \phi\]

\[= \left[ (\rho - eA)^2 + i \sigma \cdot \left[ p x p^0 - e p x A - e A x p + e^2 A x A \right] \right] \phi\]

\[= \left[ (\rho - eA)^2 - e \sigma \cdot B \right] \phi\]

\[\{ A = (0, B x, 0) \}\]

\[\nabla \cdot A = 0\]

\[B = \nabla \times A\]

\[A^2 = B^2 x^2\]

\[\text{Ansatze: } \phi_\sigma (x) = e^{i \left( p_y y + p_z z \right)} f(x) x_\sigma \]

\[= i \left( -\partial_x (B x) \right) \]

\[\text{Insert this into } 5 \text{ yields: } unit - spinor\]

\[= j \left( 0 \right) + k \left( \partial_x (B x) \right)\]

\[(\varepsilon^2-m^2) f(x) = \left( \frac{-d^2}{d x^2} + e^2 B^2 \left( x - \frac{p_y}{e B} \right)^2 \right) f(x) = \left( \varepsilon^2 - m^2 - p_z^2 + e B \sigma \right) f(x)\]

\[\text{Using algebraic manipulation,}\]

\[\frac{\partial^2}{\partial x^2} + e^2 B^2 \left( x - \frac{p_y}{e B} \right)^2 \]

\[f(x) = \left( \varepsilon^2 - m^2 - p_z^2 + e B \sigma \right) f(x)\]

Variable \( p_y = x - \frac{p_y}{e B} \). \( \kappa \omega = 2 |e| B \).

Eigenvalues \( \lambda_n = (n + 1/2) \kappa \omega = (2n + 1) |e| B \).

\[\varepsilon^2 - m^2 - p_z^2 + e B \sigma = (2n + 1) |e| B\]

\[\varepsilon^2 = (2n + 1) |e| B + m^2 + p_z^2 - e B \sigma\]

\[\varepsilon \rho \sigma = \pm \sqrt{m^2 + p_z^2 + |e| B (2n + 1) \rho \sigma} \]

Important result to be used in main discussion.
\[ \varepsilon^2 = m^2 - p_z^2 + eB \sigma = (2n+1) |e| B \]

\[ \varepsilon^2 = m^2 + p_z^2 + (2n+1) |e| B - eB \sigma \]

\[ \varepsilon^2 = m^2 + p_z^2 + |e| B (2n+1 - \sigma) \]

\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + |e| B (2n+1 - \sigma)} \]

\[ n = 2, \quad \sigma = -1 \]
\[ \varepsilon = \sqrt{m^2 + p_z^2 + 6 |e| B} \]

\[ n = 2, \quad \sigma = -1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + 6 |e| B} \]

\[ n = 0, \quad \sigma = -1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + 2 |e| B} \]

\[ n = 0, \quad \sigma = +1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + 2 |e| B} \]

\[ n = 1, \quad \sigma = +1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + 2 |e| B} \]

\[ n = 1, \quad \sigma = -1 \]
\[ \varepsilon = \mp \sqrt{m^2 + p_z^2 + 2 |e| B} \]

\[ n = 3, \quad \sigma = +1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + |e| B (6 + 1 - 1)} \]

\[ n = 3, \quad \sigma = +1 \]
\[ \varepsilon = \pm \sqrt{m^2 + p_z^2 + 6 |e| B} \]

This case is neglected.

This has no partner.
In the classical framework, the solution describes helical motion of the electron with freely chosen momentum components in $y$ and $z$ directions, but orbiting around a fixed centre:

$$x_0 = \frac{p_y}{eB}$$

Particle put in a box with dimensions $L_x, L_y, L_z$ the $y$ and $z$ motions are quantized by boundary conditions and the number of states is, $(n_l = 1)$,

$$\Delta N = \frac{L_y}{a} \frac{\Delta p_y}{\hbar} \frac{L_z}{2\pi} \frac{\Delta p_z}{\hbar},$$

$$\Delta p_y = eB \Delta x_0.$$  \text{ we sum over allowed values,}  \quad 0 < x_0 < L_x,

and obtain,

$$\Delta N = \frac{L_y}{a} \frac{|e| B L_x}{2\pi} \frac{L_z}{a} \frac{\Delta p_z}{\hbar} = \frac{|e| B}{(2\pi)^2} \Delta p_z V.$$
If we consider the EM field in isolation, it satisfies the linear Maxwell equations, and the superposition principle holds. Photon field  is described by free noninteracting Lagrange function (Lagrangian density).

\[ \mathcal{L}_0 = \frac{1}{8\pi}(E^2 - B^2). \]

This leads to linear field equations.

The vacuum of QED is a polarizable medium owing to virtual processes and obtains novel physical properties. To describe this effect, we replace \( \mathcal{L}_0 \) by an effective Lagrangian \( \mathcal{L}_{\text{eff}} \). This will contain corrections in higher orders in \( E \) and \( B \) and lead to nonlinear field equations.

In the limiting case of stationary, homogeneous EM field an exact "closed" expression can be given for \( \mathcal{L}_{\text{eff}} \).

1) W. Heisenberg, Z. Physik 38, 314 (1936)

I will be following derivation by V. Weisskopf.

EM field is characterized by two Lorentz invariant quantities,

\[ I_1 = B^2 - E^2 \quad I_2 = B \cdot E \]

Effective Lagrangian expressed as function of these invariants,

\[ \mathcal{L}_{\text{eff}} (B, E) = \mathcal{L}_{\text{eff}} (I_1, I_2) = \mathcal{L}_{\text{eff}} (B^2 - E^2, B \cdot E). \]
\[
\begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{pmatrix} =
\begin{pmatrix}
-E_1^2 - E_2^2 - E_3^2 & -E_2 B_3 - E_3 B_2 & E_1 B_3 - E_3 B_1 & -E_1 B_2 + E_2 B_1 \\
-B_3 E_2 + B_2 E_3 & -E_1^2 - B_3^2 - B_2^2 & -E_2^2 - B_3^2 - B_1^2 & -E_3^2 - B_2^2 - B_1^2
\end{pmatrix}
\]

**Contravariant matrix form**

\[
F^\mu{}^\nu =
\begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{pmatrix}
\]

**Covariant form by index lowering**

\[
F^\alpha{}^\beta{}^\eta{}^\gamma =
\begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & -B_2 & B_1 & 0
\end{pmatrix}
\]

**Faraday tensor's Hodge dual**

\[
G^{\alpha\beta} = \frac{4}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} =
\begin{pmatrix}
0 & -B_1 & -B_2 & -B_3 \\
B_1 & 0 & E_3 & -E_2 \\
B_2 & -E_3 & 0 & E_1 \\
B_3 & +E_2 & -E_1 & 0
\end{pmatrix}
\]

( Completely antisymmetric (Levi-Civita tensor) )

\[
\begin{align*}
F^\mu{}^\nu F_{\mu\nu} &= 2 (B^2 - E^2) = 2 I_1 \\
F^\mu{}^\nu \ast F_{\mu\nu} &= -4 B \cdot E = -4 I_2
\end{align*}
\]

Constructing 2 scalars by contraction of these tensors.
The Lagrangian function is gauge invariant because it depends only on the field strengths. Calculate energy $W_0$ of the vacuum per unit volume as a function of field strength.

We sum up the energy eigenvalues $\varepsilon_{p\sigma} \leq -m$ of all electrons in the Dirac sea to obtain total energy $E_0$. From this value the potential energy $U_0$ in the electric field has to be subtracted.

Energy $E_0$ contains potential energy $U_0$ of the electrons of the Dirac sea in the external field in addition to the pure energy $W_0$ of the vacuum.

We are only interested in the pure energy of the vacuum the contribution $U_0$ has to be subtracted from $E_0$:

$$E_0 = W_0 + U_0$$

$$W_0 = E_0 - U_0$$

$$U_0 = \sum_{p\sigma} \int d^3x \, \psi_{p\sigma}^{(-)} \bar{\psi}_{p\sigma}^{(-)} e^{iA_0(x)} \psi_{p\sigma}^{(-)}$$

This sum over all momenta $\mathbf{p}$ and all spin directions. Only states with negative energy $(-)$ are to be taken into account following the description of vacuum according to Dirac's hole theory picture.

$U_0$ can be expressed in terms of $E_0$ by a trick.
Let $\hat{H}(\lambda)$ be a self-adjoint Hamiltonian that depends analytically on a parameter $\lambda$ and $\psi_n(\lambda)$ a normalized eigenfunction,

$$\hat{H}(\lambda)\psi_n(\lambda) = \varepsilon_n(\lambda)\psi_n(\lambda)$$

The derivative of the energy eigenvalue w.r.t. $\lambda$ and projection onto $\langle \psi_n \rangle$,

$$\frac{d\varepsilon_n}{d\lambda} = \langle \psi_n | \frac{\partial \hat{H}}{\partial \lambda} | \psi_n \rangle + \langle \psi_n | (\hat{H} - \varepsilon_n) \frac{\partial}{\partial \lambda} | \psi_n \rangle$$

Last term is zero $\langle \psi_n | \hat{H} - \varepsilon_n \rangle = \langle \psi_n | \varepsilon_n \rangle$.

$$\frac{d\varepsilon_n}{d\lambda} = \langle \psi_n | \frac{\partial \hat{H}}{\partial \lambda} | \psi_n \rangle$$

General statement:

Potential of a stationary, homogeneous $E$ field,

$$A_0(x) = -E \cdot x$$

Use field strength as parameter $\lambda$.

$$U_0 = \vec{E} \cdot \sum_{\rho \sigma} \int d^3 x \, \psi_{\rho \sigma}^{(+)\dagger} \frac{\partial \hat{H}}{\partial E} \psi_{\rho \sigma}^{(-)} = \vec{E} \cdot \frac{\partial E_0}{\partial \vec{E}}$$

$$W_0 = E_0 - U_0 = E_0 - \vec{E} \cdot \frac{\partial E_0}{\partial \vec{E}}$$

Pure energy of the vacuum.

Total energy of the electrons in the Dirac sea.

P.E. of the $e^{-}$s of the Dirac sea in the external field.
Relationship between the energy (Hamiltonian) and Lagrangian for a system having generalised coordinates \( q_i \) reads,

\[
W = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}
\]

In Lagrangian formulation of electrodynamics, \( A \) and \( A \) play the role of generalised coordinates \( q_i \):

\[
E = -\dot{A} - \nabla A \quad \text{There is a dependence on a}
\]

\[
B = \nabla \times A \quad \text{generalised velocity (} q_i \text{) in the}
\]

Lagrangian only in the time derivative of the vector potential.

Differentiation with respect to \( \dot{A} \) is equivalent to differentiation with respect to \( E \).

\[
W = E \cdot \frac{\partial \mathcal{L}}{\partial E} - \mathcal{L}
\]

Thus we find the change of Lagrangian density of EM field is given up to a sign by the additional energy density \( E_0 \),

\[
\Delta \mathcal{L} = \mathcal{L}_0 + \mathcal{L}'
\]

\[
\mathcal{L}' = -E_0 \quad \text{(ren)}
\]

Expression \( \Delta \mathcal{L} \) still has to be renormalized. In particular, the energy of the vacuum in the absence of the EM field has to be subtracted, because it cannot be observed.
Calculating $E_0$

$$E_{p\sigma}^{(-)} = -\sqrt{m^2 + p_z^2 + |e|B(2n + 1)} \quad \sigma = \pm 1.$$  

The density of states per momentum interval is:

$$\frac{|e|B}{2\pi} \frac{dp_z}{2\pi}$$

$$\ell' = -E_0$$

$$\ell' = \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \frac{|e|B}{2\pi} \sum_{n=0}^{\infty} \sqrt{m^2 + p_z^2 + |e|B(2n + 1)}.$$

States with quantum numbers $n-1$, $\sigma = +1$ and with $n$, $\sigma = -1$ have the same energy.

$$\ell = \left( \frac{m^2 + p_z^2 + |e|B(6 + 1)}{2} \right)^{1/2}$$

- $n=1, \sigma = \pm 1$
  - $n-1=2$
  - $\sigma = \mp 1$
  - $n=3, \sigma = +1$

Only for state $n=0$, $\sigma = -1$ such a partner cannot be found.

$$n=0, \sigma = +1 \quad \ell = \sqrt{m^2 + p_z^2 + |e|B(2 + 1 - 1)}$$

All other states are doubly degenerate.
This integral is highly divergent. We can however split off a physically meaningful finite expression. First regularize eq 1 by introducing a suitably chosen cutoff factor. With the abbreviation, the function:

\[ E(n, \lambda) = \int_0^\infty dp_z \sqrt{m^2 + p_z^2 + 2|e| B_n} e^{-\lambda m^2 + p_z^2 + 2|e| B_n} \]

The regularized equation now reads:

\[ E' (\lambda) = \frac{|e| B}{\lambda^2} \left( \frac{1}{2} F(0, \lambda) + \sum_{n=1}^\infty F(n, \lambda) \right) \]

Limit \( \lambda \to 0 \) taken at end of the calculation. Physically meaningful quantities must no longer depend on \( \lambda \) then. Hence they have to approach a finite limiting value.

**Math Formula 101**

Euler Maclaurin summation formula. It is a formula for the difference between an integral and a closely related sum. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series using integrals.

\[ \sum_{i=m}^{n} f(i) = \int_{m}^{n} f(x) dx + \frac{f(m) + f(n)}{2} + \sum_{k=1}^{[p/2]} \frac{B_{2k}}{(2k)!} \left( f^{(2k-1)} \right)(m) + R_p \]

The remainder term will be neglected in our calculation.
\( f^{(k)}(x, \lambda) \) denotes the \( k \)th derivative of the function \( F(x, \lambda) \) with respect to \( x \) in our calculation.

\( B_{2n} \) are the Bernoulli numbers:
\[
B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \ldots
\]

The odd Bernoulli numbers are zero except for \( B_1 \).

The Bernoulli numbers \( B_n \) are a sequence of rational numbers. They are special values of the Bernoulli polynomials \( B_n(x) \), which occur in study of special functions like Riemann zeta function and Hurwitz zeta function.

The Bernoulli numbers are given by,
\[
B_n = B_n(0).
\]

### Bernoulli polynomials

<table>
<thead>
<tr>
<th>( B_0(x) )</th>
<th>( B_1(x) )</th>
<th>( B_2(x) )</th>
<th>( B_3(x) )</th>
<th>( B_4(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x - \frac{1}{2} )</td>
<td>( x^2 - x + \frac{1}{6} )</td>
<td>( x^3 - \frac{3}{2} x^2 + \frac{1}{2} x )</td>
<td>( x^4 - 2 x^3 + x^2 - \frac{1}{3} x )</td>
</tr>
</tbody>
</table>

Using the summation formula eq (3) can be rewritten as,

\[
\sum_{n=0}^{N} F(n, \lambda) = \frac{1}{2} F(0, \lambda) + \frac{1}{2} F(N, \lambda) + \int_{0}^{\lambda} d\eta F(n, \lambda) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( F^{(2k-1)}(N, \lambda) - F^{(2k-1)}(0, \lambda) \right).
\]

Because of eqn (2), \( F(n, \lambda) \) and all its derivatives decay exponentially at large \( n \) (for \( \lambda \neq 0 \)) so that limit \( N \to \infty \) can be taken in eqn (4),

\[
\sum_{n=0}^{\infty} F(n, \lambda) = \frac{1}{2} F(0, \lambda) + \int_{0}^{\lambda} d\eta F(n, \lambda) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} F^{(2k-1)}(0, \lambda).
\]

Eqn (5) can be rewritten as,

\[
\lambda' = \frac{1}{\pi^2} \left( \int_{0}^{\lambda} d\eta F(n, \lambda) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} F^{(2k-1)}(0, \lambda) \right).
\]
The integral in (2) defining the function $F(n, \lambda)$ can be evaluated explicitly.

$$F(n, \lambda) = \int_0^\infty dp_z \sqrt{m^2 + p_z^2 + 2|\epsilon|Bn} \ e^{-\lambda \sqrt{m^2 + p_z^2 + 2|\epsilon|Bn}} \quad (2)$$

## Substitutions

$\chi = \sqrt{p_z^2 + a^2}$

$\lambda = \sqrt{p_z^2 + a^2} \quad a^2 = m^2 + 2|\epsilon|Bn$

$$F(n, \lambda) = \int_0^\infty dp_z \sqrt{p_z^2 + a^2} \ e^{-\lambda \sqrt{p_z^2 + a^2}} \quad \chi = \sqrt{p_z^2 + a^2} \quad a^2 = \frac{p_z^2 + a^2}{a^2}$$

$$= a^2 \int_0^\infty d\chi \frac{\chi}{\sqrt{\chi^2 - 1}} \ e^{-\lambda \chi} \quad 2a^2 \chi d\chi = 2p_z dp_z \quad p_z = a^2(\chi^2 - 1)$$

$$= a^2 \left( \frac{1}{a^2} \right)^2 \frac{d^2}{d\lambda^2} \int_1^\infty d\chi \frac{e^{-\lambda \chi}}{\sqrt{\chi^2 - 1}} \quad dp_z = \frac{a^2 \chi d\chi}{p_z}$$

$$= \frac{d^2}{d\lambda^2} K_0(\lambda a) \quad p_z = a^2 \sqrt{\chi^2 - 1}$$

$$F(n, \lambda) = a^2 \frac{d^2}{d\lambda^2} K_0(z) \quad dp_z = \frac{a^2 \chi d\chi}{\sqrt{\chi^2 - 1}}$$

## Substitution

$z = \lambda a$

## Modified Bessel function of the second kind

(McDonald function)
$K_0''(z)$ evaluated by use of recursion relations for Bessel functions.

$K_0' = -K_1, \quad K_1' = K_1 \frac{1}{z} - K_2.$

$$F(n, \lambda) = a^2 \frac{d^2}{dz^2} K_0(z)$$

$$= -a^2 \frac{d}{dz} \left[ K_1(z) \right] = -a^2 \left[ \frac{1}{z} K_1(z) - K_2(z) \right]$$

$$= -a^2 \left[ \frac{1}{\lambda a} K_1(\lambda a) - K_2(\lambda a) \right]$$

$$F(n, \lambda) = -\frac{1}{\lambda^2} \left[ z K_1(z) - z^2 K_2(z) \right]$$

$$\mathcal{L}'(\lambda) = \frac{|e|B}{\pi^2} \left( \int_0^{\infty} dn F(n, \lambda) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} F(2k-1)(\lambda, \lambda) \right)$$

We need derivatives of functions in eq 3 for eq 4

w.r.t $n$. $z = \lambda a = \lambda \sqrt{m^2 + 2 |e| B n}$, $z \, d\, z = \lambda^2 |e| B \, dn$,

the $m^{th}$ derivative may be written as,

$$\left( \frac{d}{dn} \right)^m = \left( \lambda^2 |e| B \right)^m \left( \frac{1}{z} \frac{d}{dz} \right)^m.$$  

These derivatives lead to simple modified Bessel functions,

$$\left( \frac{1}{z} \frac{d}{dz} \right)^m \left( z^\nu K_\nu(z) \right) = (-1)^m z^{-\nu-m} K_{\nu-m}(z).$$

With $z (n=0) = \lambda m$, the regularized lagrangian 4 now reads,

$$\mathcal{L}'(\lambda) = -\frac{1}{\pi^2} \frac{|e|B}{\lambda^4} \int_{-1}^{1} \left( K_1(z) - z K_2(z) \right)$$

$$- \frac{|e|B}{\pi^2} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( -\frac{1}{\lambda^2} \right)^{2k-1} \left( \lambda^2 |e| B \right)^{2k-1}$$

$$\times (-1)^{2k-1} \left[ (\lambda m)^{2-2k} K_{2-2k}(\lambda m) - (\lambda m)^{3-2k} K_{3-2k}(\lambda m) \right].$$
Eq. (5) has structure of a power series in even powers of field strength $B$ multiplied by elementary change 'e'.

\[ h'(\lambda) = C_0(\lambda) + C_2(\lambda)(eB)^2 + \sum_{k=2}^{\infty} C_{2k}(\lambda)(eB)^{2k} \]

$C_0(\lambda)$ and $C_2(\lambda)$ diverge if parameter $\lambda$ approach 0.

Higher coefficients $C_4, C_6 \ldots$ are finite.

\[ C_0(\lambda) = -\frac{1}{\pi^2} \int_{\lambda_m}^{\infty} dz \int_{\lambda_m}^{\infty} dz' \frac{z^2}{z'^2} (K_1(z) - z K_2(z)) = O\left(\frac{1}{\lambda^4}\right) \rightarrow \infty. \]
\[
L'(\lambda) = -\frac{1}{\pi^2} \frac{1}{\lambda^4} \int_{\lambda m}^{\infty} dz \, z^2 \left( k_1(z) - z k_2(z) \right)
\]

\[
\frac{|e| B}{\sqrt{2}} \sum_{K=1}^{\infty} B_{2K} \left( \frac{\lambda^2 |e| B}{(2k)!} \right)^{2K-1} \left[ (\lambda m)^{2-2K} K_{2-2K}(\lambda m) \right] - (\lambda m)^{3-2K} K_{3-2K}(\lambda m) \]

**K=1 term**

\[
\frac{|e| B}{\pi^2} \frac{B_2}{2!} \left( \frac{1}{\lambda^2} \right) (e^2 |e| B)' \left[ (\lambda m)^0 K_0(\lambda m) - (\lambda m)^1 K_1(\lambda m) \right]
\]

**K=2 term**

\[
\frac{|e| B}{\pi^2} \frac{B_4}{4!} \left( \frac{1}{\lambda^2} \right) (e^2 |e| B)^2 \left[ (\lambda m)^{-2} K_{-2}(\lambda m) - (\lambda m)^{-1} K_{-1}(\lambda m) \right]
\]

Expression 1 is a power series in even powers of field strength B multiplied by the elementary charge 'e'.

\[
L'(\lambda) = C(\lambda) + C_2(\lambda) (eB)^2 + \sum_{K=2}^{\infty} C_{2K}(\lambda) (eB)^{2K}
\]

\[
C_2(\lambda) = -\frac{1}{\pi^2} \frac{B_2}{2} \left( K_0(\lambda m) - (\lambda m) K_1(\lambda m) \right)
\]

\[
C_0(\lambda) = -\frac{1}{\pi^2} \frac{1}{\lambda^4} \int_{\lambda m}^{\infty} dz \, z^2 \left( k_1(z) - zk_2(z) \right)
\]

\[C_0(\lambda) = O(\frac{1}{\lambda^4}) \to \infty.\]

Asymptotic behaviour of the Bessel function for \( z \to 0 \) is,

\[K_0(z) \to -\ln(z)\]

\[K_m(z) \to (m-1)! \frac{2^{m-1}}{z^m} \text{ for } m > 0.\]

\[C_2(\lambda) \to -\frac{1}{\pi^2} \frac{1}{12} \left( -\ln(\lambda m) - \frac{1}{\lambda m} \right)\]

\[C_2(\lambda) \to \frac{1}{\pi^2} \frac{1}{12} \ln(\lambda m)\]

**Bernoulli number**

\[B_2 = \frac{1}{6}\]

**K_1(z) \to (1-i)! \frac{z^{-1}}{2^{1-i}} \text{ for } 0 < 2^{1-i} \to \frac{1}{2}.$$
\[ B_2 = \frac{1}{6} \]

\[ \sum_{k=1}^{\infty} \frac{B_2}{2^k} \left( \frac{-1}{\lambda^k} \right) \left( X^k e^{iB} \right)^{2.1-1} \]

\[ X(-i)^{2.1-1} \left[ (\lambda m)^{2-2.1} K_{2-2.1}(\lambda m) - (\lambda m)^{3-2.1} K_{3-2.1}(\lambda m) \right] \]

\[ + \frac{|e|^2 B_2}{\pi^2} \left( \frac{1}{12} \right) \left( eB \right)^{\frac{1}{2}} \left[ K_0(\lambda m) - (\lambda m) K_1(\lambda m) \right] \]

\[ - \frac{(eB)^2}{\pi^2} \frac{B_2}{2} \left[ (K_0(\lambda m) - (\lambda m) K_1(\lambda m)) \right] \]

Asymptotic behavior of Bessel function for \( z \to 0 \) is,

\[ K_m(z) \to (m - 1)! 2^{m-1} z^{-m} \quad \text{for} \quad m > 0. \]

\[ K_0(z) \to -\ln(z). \]

\[ C_2(\lambda) = -\frac{1}{\pi^2} \frac{B_2}{2} \left[ K_0(\lambda m) - (\lambda m) K_1(\lambda m) \right] \]

\[ K_1(z) \to (1-1)! 2^{1-1} z^{-1} \quad \text{for} \quad n = 0 = \lambda m \]

\[ \to 1.2^0 \frac{1}{z} \to \frac{1}{z}. \]

\[ C_2(\lambda) \to \frac{-1}{\pi^2} \frac{1}{12} \left[ -\ln z - z \cdot \frac{1}{z} \right] \]

\[ C_2(\lambda) \to \frac{1}{\pi^2} \frac{1}{12} \ln(\lambda m) \]
Divergence problems had to occur. Energies of all states of lower continuum was summed up.

\[ C_0 \to \text{total energy of Dirac sea and as such not an observable.:} \]

Converting expression for \( C_0 \) into 3D momentum integral:

\[ C_0(\lambda) = \frac{1}{\pi^2} \int_0^\infty dn \ F(n, \lambda). \]

\[ F(n, \lambda) = \int_0^\infty dp_z \sqrt{m^2 + p_z^2 + 2|e|Bn} e^{-\lambda\sqrt{m^2 + p_z^2 + 2|e|Bn}} \]

\[ C_0(\lambda) = \frac{1}{\pi^2} \int_0^\infty dn \int_0^\infty dp_z \sqrt{m^2 + p_z^2 + 2|e|Bn} e^{-\lambda\sqrt{m^2 + p_z^2 + 2|e|Bn}} \]

Cylindrical coordinates:

\[ p_\perp = 2|e|Bn \]
\[ 2p_\perp dp_\perp d\phi = 2|e|B dn \]

\[ dn = \frac{p_\perp dp_\perp d\phi}{|e|B} \]

\[ d^2p_\perp = p_\perp dp_\perp d\phi \]

Factor of \( \frac{1}{2} \) for change of limit of integration.

No \( \phi \) dependence of momentum values so additional \( 2\pi \) added in denominator.

This is just the regularized expression for the negative of the energy of the lower continuum in the absence of an external field. We need to subtract \( C_0 \) in eqn to obtain a meaningful expression.

\[ \lambda'(\lambda) = C_0(\lambda) + C_{2z}(\lambda)(eB)^2 + \sum_{k=2}^\infty C_{2k}(\lambda)(eB)^{2k} \]
$C_2 e^2 B^2$ has exactly form of free Lagrangian.

$$\text{left} = \mathcal{L}_0 + \mathcal{L}' - \mathcal{L}_0$$

$$= \mathcal{L}_0 + C_2 (eB)^2 + [\mathcal{L}' - \mathcal{L}_0 - C_2 (eB)^2]$$

$$= -(1 - 8\pi C_2 e^2 B^2) + \sum_{n=2}^{\infty} C_{2n} (eB)^{2n}$$
\[ \mathcal{L}_0 = \frac{1}{8\pi} (E^2 - B^2). \]

\[ E = 0 \]

\[ \mathcal{L}_0 = -\frac{1}{8\pi} B^2. \]

\[ \mathcal{L}' = c_0 + c_2 (eB)^2 + \sum_{k=2}^{\infty} c_{2k} (eB)^{2k} \]

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}' - c_0. \]

\[ c_0 = \mathcal{L}' - c_2 (eB)^2 - \sum_{k=2}^{\infty} c_{2k} (eB)^{2k}. \]

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_0 + c_0 + c_2 (eB)^2 + \sum_{k=2}^{\infty} c_{2k} (eB)^{2k} \]

\[ \mathcal{L}' - c_0 - c_2 (eB)^2 = \sum_{k=2}^{\infty} c_{2k} (eB)^{2k}. \]

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_0 + c_2 (eB)^2 + \left[ \mathcal{L}' - c_0 - c_2 (eB)^2 \right]. \]

\[ = -\frac{B^2}{8\pi} - c_2 (eB)^2 + \sum_{n=2}^{\infty} c_{2n} (eB)^{2n} \]

\[ \mathcal{L}_{\text{eff}} = -\left(1 - 8\pi e^2 C_2^2\right) \frac{B^2}{8\pi} + \sum_{n=2}^{\infty} c_{2n} (eB)^{2n} \]

Free Lagrangian is multiplied by a constant \((1 - 8\pi e^2 C_2)^2\). Presence of this factor cannot be observed physically.

Constant factor only leads to a redefinition of the field strength and charge.

Renormalized elementary charge,

\[ e_R = \frac{e}{\sqrt{1 - 8\pi e^2 C_2}}. \]

Renormalized field strength

\[ B_R = \frac{e}{e_R} B = \sqrt{1 - 8\pi e^2 C_2} B. \]

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}' \]

\[ e_R, B_R \text{ are physically observable quantities.} \]
$K_m(z) \rightarrow (m-1)! 2^{m-1} z^{-m}$ for $m > 0$.

\[ \mathcal{L}' = -\frac{1}{\kappa^2} \frac{1}{\lambda^4} \int d\lambda \zeta z^2 \left( K_1(z) - z K_2(z) \right) \]

\[ = \frac{1}{\pi^2} B_{2k} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \left( \frac{\lambda^2 |\epsilon| B}{\lambda^2} \right)^{2k-1} \]

\[ x(-1)^{2k-1} \left( (\lambda m)^{2-2k} K_{2k-2}(\lambda m) - (\lambda m)^{3-2k} k_{3-2k}(\lambda m) \right) \]

\[ (K_{-n}(z)) = k_m(z) \quad \text{Renormalized correction to Lagrangian,} \]

\[ \mathcal{L}' = -\frac{1}{\pi^2} \lim_{\lambda \rightarrow 0} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\lambda^2 |\epsilon| B}{\lambda^4} \right)^{2k} \]

\[ x \left[ (\lambda m)^{2-2k} K_{2k-2}(\lambda m) - (\lambda m)^{3-2k} k_{3-2k}(\lambda m) \right] \]

\[ = -\frac{1}{\pi^2} \lim_{\lambda \rightarrow 0} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \lambda^{4-4k} (\epsilon B)^{2k} (\lambda m)^{2-2k} (2k-3)! 2^{-2k+2} (\lambda m)^{2k-3} \]

\[ = -\frac{1}{\pi^2} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} m^{4-4k} (\epsilon B)^{2k} 2 (2k-3)! \]

\[ \mathcal{L}' = -\frac{1}{\pi^2} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{2 \epsilon B}{8} \right)^k m^{4-4k} (2k-3)! \]

\[ \mathcal{L}' = -\frac{1}{8 \pi^2} \sum_{k=2}^{\infty} \left( \epsilon B \right)^k B_{2k} m^{4-4k} \frac{\Gamma(2k-2)}{(2k)!} \]

\[ \Gamma(n) = (n-1)! \]

\[ K_{2k-2}(\lambda m) \rightarrow (2k-2-1)! 2^{2k-2-1} (\lambda m)^{2-2k} \]

\[ \rightarrow (2k-3)! 2^{2k-3} (\lambda m)^{2-2k} \]
The renormalized correction $\lambda'$ to the lagrangian is,

$$
\lambda' = -\frac{1}{\pi^2} \lim_{\lambda \to 0} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \left( \lambda^2 |eB| \right)^{2k} \frac{1}{\lambda^4 (\lambda m)^{2-2k}} K_{2k-2}(\lambda m) \\
- (\lambda m)^{3-2k} K_{2k-3}(\lambda m)
$$

**Case $k=2$**

$$
(\lambda m)^{2-2k} K_{2k-2}(\lambda m) - (\lambda m)^{3-2k} K_{2k-3}(\lambda m)
$$

$$
(\lambda m)^{2} K_{2}(\lambda m) - (\lambda m)^{3} K_{1}(\lambda m)
$$

$$
K_{-n}(z) = K_{n}(\bar{z})
$$

$$
\lambda'^{4k-4} (\lambda m)^{2-2k} (\lambda m)^{-2k+2}
$$

$$
\lambda'^{4k-4} \cdot \lambda^{2-2k} \cdot \lambda^{-2k+2} m^{-2k} m^{-2k+2}
$$

$$
\lambda'^{2k} k^{2k} \cdot 2^{-3} (\lambda m)^{2-2k} (2k-3) ! (2k-3)^{2k} \lambda^{2k-3} (\lambda m)^{-2k+2}
$$

$$
\frac{1}{8} (2eB)^{2k}
$$

$$
\lambda' = -\frac{1}{\pi^2} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{1}{2eB} \right)^{2k} m^{-4k} (2k-3) !
$$

$$
\lambda' = -\frac{1}{8 \pi^2} \sum_{k=2}^{\infty} \left( \frac{2eB}{B_{2k}} \right)^{2k} m^{-4k} \frac{T'(2k-2)}{(2k)!}
$$

**Gamma function is a generalisation of the factorial function.**

$$
T'(n) = (n-1)!
$$

Using the integral representation of the gamma function,

$$
T'(z) = \int_{0}^{\infty} e^{-\eta} \eta^{z-1} d\eta
$$

**Case $k=2$**

$$
T'(2,2-3) ! = (4-3) ! = 1 !
$$

$$
T'(2,2-2) ! = (2,2) !
$$

$$
T'(2) = 4 !
$$
Using this representation eq. (10) can be rewritten as

\[ \kappa' = -\frac{1}{8\pi^2} \sum_{k=2}^{\infty} (2\,|e|B)^{2k} B_{2k} \, m^{1-4k-1} \int_0^\infty d\eta \, e^{-\eta} \eta^{2k-3} \]

\[ \kappa' = -\frac{1}{8\pi^2} |e| B m^3 \int_0^\infty d\eta \, e^{-\eta} \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \frac{|e|B\eta}{m^2} \right)^{2k-1} \]

Careful inspection of the series in this expression it is identical to the Taylor expansion of the hyperbolic cotangent function,

\[ \coth x = \frac{1}{x} + \frac{x}{3} + \sum_{k=2}^{\infty} \frac{2}{(2k)!} B_{2k} x^{2k-1} \]

\[ \coth x = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2 x^5}{945} + \ldots \]

\[ \coth x = x^{-1} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1}, \quad 0 < |x| < \pi. \]

Introducing dimensionless field strength,

\[ \tilde{B} = \frac{B}{B_{cr}} = \frac{|e|}{m^2} B. \]

Critical magnetic field,

\[ B_{cr} = \frac{m^2}{|e|} = \frac{m^2 c^3}{|e| |e|^2} = 4.4 \times 10^{18} \text{ Gauss} = 4.4 \times 10^9 \text{ Tesla}. \]
\[ \kappa' = -\frac{1}{8\pi^2} \left| e/\beta m^2 \right| \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^2} \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \frac{\left( le/\beta \eta \right)^{2k-1}}{m^2} \]

\[ \tilde{B} = \frac{B}{B_{cr}} = \frac{le}{m^2} B. \]

\[ \kappa' = -\frac{1}{8\pi^2} \left| e/\beta m^2 \right| \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^2} \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} \]

\[ = -\frac{1}{8\pi^2} \left| e/\beta m^2 \right| \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} \]

\[ = -\frac{1}{8\pi^2} m^4 \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} \]

\[ \cot h (x) = \frac{1}{x} + \frac{x}{3} + \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \frac{x^{2k-1}}{x} \]

\[ \coth (\tilde{B} \eta) = \frac{1}{\tilde{B} \eta} + \frac{\tilde{B} \eta}{3} + \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} \]

\[ \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} = \coth (\tilde{B} \eta) - \frac{1}{\tilde{B} \eta} - \frac{\tilde{B} \eta}{3} \]

\[ \left\{ \left( \tilde{B} \eta \right) \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \left( \tilde{B} \eta \right)^{2k-1} = \left( \tilde{B} \eta \right) \coth (\tilde{B} \eta) - 1 - \frac{1}{3} \left( \tilde{B} \eta \right)^2 \right\} \]

\[ \kappa' = \frac{m^4}{8\pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left( -\tilde{B} \eta \cot h (\tilde{B} \eta) + 1 + \frac{1}{3} \left( \tilde{B} \eta \right)^2 \right). \]
Pure magnetic field case

\[ \mathcal{L}'(E = 0, B) = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left( -B \eta \coth(\tilde{B} \eta) + 1 + \frac{1}{3} (\tilde{B} \eta)^2 \right). \]

Pure electric field \((B = 0)\), \(B^2 - E^2\), \(B, E\) express result in terms of invariants.

\[ \mathcal{L}'(E, B = 0) = \mathcal{L}'(I_1 = 0 - E^2, I_2 = 0) \]

= \mathcal{L}'(I_1 = (iE)^2, I_2 = 0)

= \mathcal{L}'(E = 0, B = iE)

One can use solution of pure magnetic field and replace \(B\) by \(iE\).

\[ \coth(ix) = -i \text{cat}(x) \]

\[ \mathcal{L}'(E, B = 0) = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \eta^2 \frac{e^{-\eta}}{\tilde{E} \eta} \left( -(iE) \eta \coth(iE \eta) + 1 + \frac{1}{3} (iE \eta)^2 \right). \]

\[ = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \eta^2 \frac{e^{-\eta}}{\tilde{E} \eta} \left( -iE \eta \coth(iE \eta) + 1 + \frac{1}{3} (iE \eta)^2 \right). \]

\(\tilde{E} = E / E_{ct} = \frac{ie}{m^2} E\).

Heisenberg calculated the general result in 1936 which I quote here for the case of constant electric and magnetic fields.

\[ \mathcal{L}'(B \parallel E) = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\tilde{E} \eta} \left[ -\tilde{E} \eta \coth(\tilde{E} \eta) \tilde{B} \eta \coth(\tilde{B} \eta) + 1 - \frac{1}{3} (\tilde{E}^2 - \tilde{B}^2) \eta^2 \right]. \]
Taylor expansion of the hyperbolic cotangent function,

$$\coth(x) = \frac{1}{x} + \frac{x}{3} + \sum_{k=2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} x^{2k-1}.$$ 

Taylor expansion up to 3rd term,

$$\coth(x) = \frac{1}{x} + \frac{x}{3} + \frac{2^4}{4!} B_4 x^3$$

$$= \frac{1}{x} + \frac{x}{3} + \frac{16}{24} \left( \frac{-1}{30} \right) x^3$$

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{1}{45} x^3$$

$$k' (\tilde{B}|E) = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left[ -\tilde{E} \eta \coth(\tilde{E} \eta) \tilde{B} \eta \coth(\tilde{B} \eta) + 1 - \frac{1}{3} (\tilde{E}^2 - \tilde{B}^2) \eta^2 \right]$$

Case of both constant electric and magnetic fields.

\#) limiting case of weak fields \( \tilde{E} \ll 1, \tilde{B} \ll 1 \).

$$\coth(\tilde{E} \eta) = \frac{1}{\tilde{E} \eta} + \frac{\tilde{E} \eta}{3} - \frac{1}{45} (\tilde{E} \eta)^3$$

$$\coth(\tilde{B} \eta) = \frac{1}{\tilde{B} \eta} + \frac{\tilde{B} \eta}{3} - \frac{1}{45} (\tilde{B} \eta)^3$$

$$-\tilde{E} \eta \coth(\tilde{E} \eta) = -1 - \frac{\tilde{E} \eta}{3} + \frac{1}{45} (\tilde{E} \eta)^4$$

$$\tilde{B} \eta \coth(\tilde{B} \eta) = 1 + \frac{\tilde{B} \eta}{3} - \frac{1}{45} (\tilde{B} \eta)^4.$$
\[ \lambda' = \frac{m^4}{8\pi^2} \cdot \frac{1}{45} \int_0^\infty d\eta \ \eta \ e^{-\eta} \left( \tilde{E}^4 + \tilde{B}^4 + 5 \tilde{E}^2 \tilde{B}^2 \right) \]

\[ \lambda' = \frac{m^4}{8\pi^2} \cdot \frac{1}{45} \left( \frac{|\eta|}{m^8} \tilde{E}^4 + \frac{|\eta|}{m^8} \tilde{B}^4 + 5 \frac{|\eta|}{m^4} \tilde{E}^2 \tilde{B}^2 \right) \]

\[ \lambda' = \frac{m^4}{8\pi^2} \cdot \frac{1}{45} \left( \frac{|\eta|}{m^8} E^4 + \frac{|\eta|}{m^8} B^4 + 5 \frac{|\eta|}{m^4} E^2 B^2 \right) \]

\[ \lambda' = \frac{m^4 \cdot e^4}{8\pi^2} \cdot \frac{1}{45} \left( (B^2 - E^2)^2 + 7(E \cdot B)^2 \right) \]

This result is valid in every frame of reference because it was expressed in terms of the invariants \( I_1 \) and \( I_2 \).

\[ \int_0^\infty dx \ x \ e^{-x} \]

\[ = \left[ svd \right] - \int \left[ \frac{du}{dx} \cdot svd \right] dx \]

\[ = x e^{-x} dx - \int [1] e^{-x} dx \]

\[ = -xe^{-x} + \int e^{-x} dx \]

\[ = -xe^{-x} - e^{-x} \bigg|_0^\infty \]

\[ = 1. \]
\[ \tilde{B} = \frac{B}{B_{cr}} \gg 1. \]

**Strong magnetic fields, i.e., \( \tilde{B} \gg 1 \).**

\[ \lambda'(E = 0, \tilde{B}) = \frac{m^4}{8 \pi^2} \int_0^\infty d\eta \, \frac{e^{-\eta}}{\eta^3} \left( -\tilde{B} \eta \coth(\tilde{B} \eta) + 1 + \frac{1}{3} (\tilde{B} \eta)^2 \right). \]

\[ \tilde{\gamma} = \frac{\eta}{\tilde{B}}, \quad \eta = \frac{\tilde{\gamma}}{\tilde{B}}, \quad d\tilde{\gamma} = \tilde{B} d\eta. \]

\[ d\eta = d\tilde{\gamma}/\tilde{B}. \]

\[ \lambda'(E = 0, \tilde{B}) = \frac{m^4}{8 \pi^2} \int_0^\infty d\tilde{\gamma} \, \frac{e^{-\tilde{\gamma} / \tilde{B}}}{\tilde{\gamma}^3} \tilde{B}^3 \left( -\tilde{\gamma} \coth(\tilde{\gamma}) + 1 + \frac{1}{3} (\tilde{\gamma})^2 \right). \]

\[ \lambda'(E = 0, \tilde{B}) = \frac{m^4 \tilde{B}^2}{8 \pi^2} \int_0^\infty d\tilde{\gamma} \, \frac{e^{-\tilde{\gamma} / \tilde{B}}}{\tilde{\gamma}^3} \left( \frac{1}{3} \tilde{\gamma}^2 + 1 - \tilde{\gamma} \coth(\tilde{\gamma}) \right). \]

\[ \lambda'(E = 0, \tilde{B}) = \frac{m^4 \tilde{B}^2}{8 \pi^2} \int_0^\infty d\tilde{\gamma} \, \frac{e^{-\tilde{\gamma} / \tilde{B}}}{\tilde{\gamma}} \left( \frac{1}{3} + 1 - \tilde{\gamma} \coth(\tilde{\gamma}) \right). \]

For \( \tilde{\gamma} \ll 1 \) integrand is attenuated by expression in parentheses (because \( \coth \tilde{\gamma} = 1/\tilde{\gamma} + \tilde{\gamma}/3 - \tilde{\gamma}^3/45 + \ldots \)) and for large \( \tilde{\gamma} \gg \tilde{B} \) by the exponential factor. Reasonable approximation to replace the integration bounds by those values and further neglect the variation of the second term in parentheses and of \( \exp(-\tilde{\gamma} / \tilde{B}) \) in this range.

\[ \lambda' \approx \frac{m^4 \tilde{B}^2}{8 \pi^2} \int_0^\infty d\tilde{\gamma} \, \frac{1}{\tilde{\gamma}^3} = \frac{m^4 \tilde{B}^2}{24 \pi^2} \ln(\tilde{B}). \]
\[
\lambda' = \frac{m^4 B'^2}{2 \pi^2} \ln \left( \frac{B}{B_c} \right) \\
\bar{B} = \frac{B}{B_c} = \frac{1/16}{m^2} B
\]

\[
\lambda_0 = -\frac{1}{8 \pi} B^2
\]

\[
\lambda' = \frac{m^4 \ell e B^2}{24 \pi^2 m^4} \ln \left( \frac{16 \ell B}{m^2} \right)
\]

\[
\lambda_0 = \frac{e B^2}{3 \pi} \ln \left( \frac{16 \ell B}{m^2} \right)
\]

\[
= \frac{e B^2}{3 \pi} \ln \left( \frac{16 \ell B}{m^2} \right)
\]

For \( \lambda' = \lambda_0 \), one has to reach entirely unrealistic field strengths,

\[
\frac{3 \pi}{e^2} = \ln \left( \frac{16 \ell B}{m^2} \right)
\]

\[
\exp \frac{3 \pi}{\alpha} = \frac{16 \ell B}{m^2} = B
\]

\[
B = B_c \exp \left( \frac{3 \pi}{\alpha} \right) = 10 \ B_c
\]

Due to small EM coupling constant.
\( \mathcal{L}' = \frac{m^4 \widetilde{B}^2}{8 \pi^2} \int \frac{d \zeta}{\zeta} \frac{1}{3} = \frac{m^4 \widetilde{B}^2}{24 \pi^2} \ln(\widetilde{B}). \)

\[ \mathcal{L}_0 = \frac{1}{8 \pi} (E^2 - B^2) \quad (E = 0 ; B) \]

\[ \mathcal{L}_0 = - \frac{1}{8 \pi} B^2. \]

\[ \frac{\mathcal{L}'}{\mathcal{L}_0} = \frac{m^4 \widetilde{B}^2}{24 \pi^2} \ln(\widetilde{B}) \times \frac{8 \pi}{B^2} \]

\[ = \frac{m^4 |e|^2}{24 \pi^2 m^4} \ln \left( \frac{|e| m^2 B}{B^2} \right) \times \frac{8 \pi}{B^2} \]

\[ \frac{\mathcal{L}'}{\mathcal{L}_0} = \frac{e^2}{3 \pi} \ln \left( \frac{|e| m B}{m^2} \right) \]

Comparing this to the free Lagrangian \( \mathcal{L}_0 \), we see nonlinear effects stay small in QED.

Inverse of logarithmic function is the exponential function.

\[ \frac{\mathcal{L}'}{\mathcal{L}_0} = \frac{e^2}{3 \pi} \ln \left( \frac{m B}{e^2} \right) = \frac{e^2}{3 \pi} \ln \left( \frac{B}{B_{cr}} \right) \]

In order to have \( \mathcal{L}' = \mathcal{L}_0 \) one would have to reach unrealistic field strengths,

\[ 4 = \frac{e^2}{3 \pi} \ln \left( \frac{B}{B_{cr}} \right) \]

\[ \frac{3 \pi}{e^2} = \ln \left( \frac{B}{B_{cr}} \right) \]

\[ \exp \left( \frac{3 \pi}{e^2} \right) = \frac{B}{B_{cr}} \]

\[ \theta = \theta_{cr} e^{3\pi/\alpha} = 10^{560} B_{cr}. \]

Sommerfeld’s fine structure constant,

\[ \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \]

When we treat the electric charge as the fundamental quantity and not the magnetic charge (monopole).

This is due to the smallness of the electromagnetic coupling constant.
Schwinger effect

Lagrangian of strong electric fields:

\[ L'(E, B = 0) = \frac{m^4}{8\pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left[ -\tilde{E}_n \cot(\tilde{E}_n) + 1 - \frac{1}{3}(\tilde{E}_n)^2 \right] \]

\[ \tilde{E} = \eta \tilde{E} \quad \eta = \frac{\tilde{E}}{E} \]

\[ d\tilde{E} = \tilde{E} d\eta \quad d\eta = d\tilde{E}/\tilde{E} \]

\[ L'(E, B = 0) = \frac{m^4}{8\pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left[ +\tilde{E}_n \cot(\tilde{E}_n) \eta + \frac{1}{3}(\tilde{E}_n)^2 \right] \]

\[ L'(E, B = 0) = \frac{m^4 \tilde{E}^2}{8\pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left( \tilde{E}_n \cot(\tilde{E}_n) + \frac{1}{3}(\tilde{E}_n)^2 \right) \]

This result is not well defined because cotangent has poles on real axis. Integral for energy density:

\[ E_0 = \frac{m^4 \tilde{E}^2}{8\pi^2} \int_0^\infty d\eta \frac{e^{-\eta}}{\eta^3} \left( \tilde{E}_n \cot(\tilde{E}_n) + \frac{1}{3}(\tilde{E}_n)^2 \right) \]

can be given a value by choosing a contour in complex plane.

Energy obtains a negative imaginary part. Cauchy residue theorem calculate magnitude by taking half of negative residuum at each pole.

Deformed integration contour chosen.
\[ \text{Im} \Phi_0^2 = -\frac{1}{\hbar} 2\pi i \sum_{n=1}^{\infty} \text{Res} \left[ \frac{e^{-n\pi \sqrt{E^2}}}{(n\pi)^3} \right] \]

\[ = \frac{1}{\hbar} 2\pi i \sum_{n=1}^{\infty} \frac{e^{-n\pi \sqrt{E^2}}}{(n\pi)^3} \]

Complex energies characterize decay of a quantum mechanical state.

Time dependent state \( |\phi(t)\rangle = e^{-iEt} |\phi\rangle \).

Its probability is,

\[ P(t) = \langle \phi(t) | \phi(t) \rangle = e^{-i(E-E^*)t} \langle \phi | \phi \rangle \]

\[ = e^{-2\text{Im}(Et)} \]

In this discussion vacuum state which is originally free of particles decays spontaneously in strong electric field by creation of \( e^-e^+ \) pairs. Particle creation rate per unit volume and time is,

\[ w = 2 \text{Im}(\Delta^I) \exp(-\Delta) = \frac{1}{4\pi^2} \frac{(mc^2)^2}{\hbar} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left( -n^2 \frac{m^2 c^4}{4\hbar E} \right) \]

This has essential singularity in limit \( e \to 0 \).

Thus pair creation in a strong field is a nonperturbative effect which cannot be calculated by series expansion in coupling constant.