QFT curved spacetime seminar 2/9/2018

First a note:
Digesh used the following action

$$\mathcal{L} = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right) \qquad (1)$$

for which

$$\mathcal{L}' = \frac{1}{2}|g|^{1/2}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right)$$ i.e. $\xi = 0$

The reason to begin with (1) is that when

$$\mathcal{L} \supset |g|^{1/2}\lambda\phi^4$$

that is we consider an interacting scalar field Then it will turn out that $|g|^{1/2}\xi R\phi^2$ is needed for the renormalization procedure.

Reminders to my self.
note that

$$\langle f_1|f_2\rangle = i\int dV_x\left[f_1^*(x,t)\,\partial_0 f_2(x,t) - \partial_0 f_1^*(x,t)\, f_2(x,t)\right]$$

$$\equiv i\int dV_x\, f_1^*\overleftrightarrow{\partial_0} f_2$$

? $\boxed{\langle f_1|f_2\rangle^* = \langle f_2|f_1\rangle = -\langle f_1^*|f_2^*\rangle}$

and $\boxed{f_1^*\overleftrightarrow{\partial_\nu} f_2 \equiv f_1^*\partial_\nu f_2 - \partial_\nu f_1^*\, f_2}$

$$\langle f_1|f_2\rangle = i\int d^{n-1}x\, |g|^{\frac{1}{2}} g^{0\mu} f_1^*\overleftrightarrow{\partial_\nu} f_2$$

note further that digesh made the following points when discussing the metric

$$ds^2 = dt^2 - a(t)(dx_i dx^i) \quad (1)$$

i) The creation of particles by the changing metric

ii) The role of "Bogolubov transformations" or linear transformations of a^+ & a in describing this particle creation

iii) The ambiguity of the vacuum.

in todays talk i would like to do two things.

1.) There is an additional insight to be gained by studying (1)

"spin statistics follows from the requirement of consistent dynamic evolution of the field equations."

2.) Discuss the probability distribution of the created particles.

(if there is time (doubtful) i would also like to give an exactly soluble system, that is a particular form of $a(t)$ for which $\left|\frac{\beta_k}{\alpha_k}\right|$ can be given)

1.) spin statistics and dynamics

as a reminder
The equation of motion for our system is still

$$\Box\phi = 0 = a^{-3}\partial_t(a^3\partial_t\phi) - a^{-2}\partial_i\partial^i\phi = 0 \quad \#$$

We take $d\tau = a^{-3}dt$ or $\partial_\tau = a^3\partial_t$

for which # becomes

$$a^{-6}\partial_\tau^2\phi - a^{-2}\partial_i\partial^i\phi = 0$$

$$\rightarrow \partial_\tau^2\phi - a^4\partial_i\partial^i\phi = 0$$

furthermore ϕ is expanded in a set of mode functions $\phi = \sum_k A_k f_k + A^+ f_k^*$

we take $f_k = V^{-1/2} e^{ik\cdot x}\varphi_k(\tau)$ (big box normalization)

so that $k^i = \frac{2\pi n^i}{L}$ and $k = |\vec{k}|$

$\therefore$

$$V^{-1/2}e^{ik\cdot x}\partial_\tau^2\varphi_k - a^4V^{-1/2}(i^2k^2)e^{ikx}\varphi_k = 0$$

$$\partial_\tau^2\varphi_k + a^4k^2\varphi_k = 0$$

$$\rightarrow \boxed{\varphi_k \cong B_1 e^{a^2k\tau i} + B' e^{-a^2k\tau i}}$$

initial condition

$a(t) = a_1$ $\quad t \rightarrow -\infty$ $\quad$ then $f_k = \tilde{V}^{-1/2}(2\tilde{\omega}_k)^{-1/2}e^{-i\tilde{\omega}_k\tau}$

with $\tilde{\omega}_k = \omega_{k1}a^3$, $\tilde{k} = k/a = \omega_{k1}$

then $\hat{V} = Va^3$

it can be shown

$[A_k, A_{k'}^+] = \delta_{kk'}$, create a particle with A_k^+

with P: k/a

E: ω_{k1}

Morale of the story:

- We have quantized ϕ at early times
  Creation: A_k^+
  Annihilation: A_k

Let $|0\rangle$ be the vacuum for these early times

$$A_k|0\rangle = 0 \quad \forall \vec{k}$$

Now at late times again we have

$$(\partial_\tau^2 + a^4 k^2)\psi(\tau) = 0$$

$*$ $\psi_k^\pm \sim (2a^3\omega_{2k})^{-1/2} e^{\mp i\omega_{2k}a^3\tau}$, $\omega_{2k} = k/a_2$

and

$$\psi_k = \alpha_k \psi_k^+ + \beta_k \psi_k^- \quad \text{for } t \to \infty \text{ with } a_2 = a(t)$$

Computing the Wronskian

$*'$ $\psi_k \partial_\tau \psi_k^* - \psi_k^* \partial_\tau \psi_k = i$ ← determined by $a(t)$ at $t \to -\infty$

$\Rightarrow |\alpha_k|^2 - |\beta_k|^2 = 1$ (Plug $*$ into $*'$)

At late times

$$\phi = \sum_k a_k g_k(x) + a_k^+ g_k^*(x)$$

$$\Rightarrow \quad \tilde{g}_k \cong (Va_3^3)^{-1/2}(2\omega_{2k})^{-1/2} e^{i(kx - \omega_{2k}t)}$$

and $a_k = \alpha_k A_k + \beta_k^* A_{-k}^+$

Please note suppose we are in a comoving frame during a time of rapid change of $a(t)$

if $\Delta t \ll 1$ Then a large # of particles is created due to time-energy uncertainty

if $\Delta t \gg 1$ then a large # of particles is created due to $a(t)$.

There is no time Δt for which the minimum uncertainty in the # of particles is zero. $\therefore$ N is not a well defined op. for this period.

Morale of the story:

- we have now quantized the field at late times

annihilation: $a_k = \alpha_k A_k + \beta^* A^+_{-k}$

creation: $a^+_k = \ldots$

- during Periods of rapid $a(t)$ change particle # is not a well defined notion.

Finally... After that review.

There is a curious relation that appears in curved spacetime not present in flat space.

We will show that only bose-Einstien statistics is consistent with the dynamics of a spin-0 field in a curved space time.

- first restore $m \neq 0$; $\xi \neq 0$ So that the equations of motion are

$$(\Box + m^2 + \xi R)\phi = 0 \quad [ds^2 = dt^2 - a(t)(dx_i dx^i)]$$

$$R = 6\left(\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)\right)$$

[if you took cosmology get out your notes or Head to a GR text or Better yet get out mathematica with diff.geo and show this.]

$$\rightarrow \quad a^{-3}\partial_t(a^3\partial_t\phi) - a^{-2}\partial_i\partial^i\phi + (m^2 + \xi R)\phi = 0$$

and that

$$a(t) = \begin{cases} a_1 & t \to -\infty \\ a_2 & t \to \infty \end{cases}$$

as before we write

$$\phi = \sum_k A_k f_k - A^+_k f^*_k$$

Where again $f_k \sim (Va_1^3)^{-1/2}(2\omega_{1k})^{-1/2} e^{i(kx - \omega_{1k}t)}$

is The positive frequency minkowski Solution.

However now $\omega_{1k} = \sqrt{k^2 + m^2}$ $\quad k \equiv \kappa_1/a_1$

Why no scalar curvature contributions?
We are considering $a(t) = \begin{cases} a_1 \in \mathbb{R} & t \to -\infty \\ a_2 \in \mathbb{R} & t \to \infty \end{cases}$

This means $R(g_{\mu\nu})\Big|_{t\to-\infty} = 6\left(\left(\frac{1}{a_1}\frac{d}{dt}(a_1)\right)^2 - \frac{1}{a_1}\frac{d^2}{dt^2}(a_1)\right)$

$= 0 = R(g_{\mu\nu})\Big|_{t\to\infty}$

now again at late times

$$\phi = \sum_k a_k g_k - a_k^\dagger g_k^*$$

Where at late times g reduces to the positive ω Minkowski solution

$$g_k \sim (Va_2)^{-1/2}(2\omega_{2k})^{-1/2} e^{i(kx - \omega_{2k} t)}$$

$k \equiv \kappa_1/a_2$

$\omega_{2k} = \sqrt{k^2 + m^2}$

Since the field equations imply that the late time behavior of $f_k \sim (Va_2)^{-1/2}(2\omega_{2k})^{-1/2}\left(\alpha_k e^{-i(\vec{k}x + \omega t)} + \beta e^{-i(kx - \omega t)}\right)$

$k \equiv \kappa_1/a_2 \quad \omega \equiv \omega_{2k}$

The late time annihilation operator is related to the early time one via Bogolubov transformation

$$a_k = \alpha_k A_k + \beta_k^* A_{-k}^\dagger$$

Furthermore

$$\langle f_1 | f_1 \rangle = 1 = |\alpha_k|^2 - |\beta_k|^2$$

it should be noted that the condition

$\langle f_k | f_k \rangle = 1 = |\alpha_k|^2 - |\beta_k^2|$

is purely the result of the field equations. it has nothing to do with the canonical commutation relations which we now consider

$$[A_{\vec{k}}, A_{\vec{k}'}]_\pm = [A^+_{\vec{k}}, A^+_{\vec{k}'}]_\pm = 0$$

$$[A_{\vec{k}}, A^+_{\vec{k}'}]_\pm = \delta_{kk'}$$

here $[a, b]_+ = ab + ba$ - Fermi-Dirac

$[a, b]_- = ab - ba$ - Bose-Einstien

$$[a_{\vec{k}}, a^+_{\vec{k}'}]_\pm = [\alpha_k A_{\vec{k}} + \beta^*_k A^+_{-\vec{k}}, \alpha^*_{k'} A^+_{\vec{k}'} + \beta_{k'} A_{-\vec{k}'}]$$

$$= \alpha_k \alpha^*_{k'} [A_{\vec{k}}, A^+_{\vec{k}'}]_\pm \pm \beta^*_k \beta_{k'} [A_{-\vec{k}'}, A^+_{-\vec{k}}]_\pm$$

$$= |\alpha_k| \delta_{\vec{k},\vec{k}'} \pm |\beta_k| \delta_{-\vec{k}',-\vec{k}}$$

$[a_{\vec{k}}, a^+_{\vec{k}'}] = (|\alpha_k| \pm |\beta_k|) \delta_{\vec{k},\vec{k}'}$ note $\delta_{-\vec{k},-\vec{k}'} = \delta_{\vec{k},\vec{k}'}$

This is Good however what about $[a_k, a_{k'}]$?

$$[a_{\vec{k}}, a_{\vec{k}'}]_\pm = [\alpha_k A_{\vec{k}} + \beta^*_k A^+_{-\vec{k}}, \alpha_{k'} A_{\vec{k}'} + \beta^*_{k'} A^+_{-\vec{k}'}]_\pm$$

$$= \alpha_k \beta^*_{k'} [A_{\vec{k}}, A^+_{-\vec{k}'}] \pm \beta^*_k \alpha_{k'} [A_{\vec{k}'}, A^+_{-\vec{k}}]_\pm$$

$$= \alpha_k \beta^*_{k'} \delta_{\vec{k},-\vec{k}'} \pm \beta^*_k \alpha_{k'} \delta_{\vec{k}',-\vec{k}}$$

$$= (\alpha_k \beta^*_{k'} \pm \alpha_{k'} \beta^*_k) \delta_{\vec{k}',-\vec{k}}$$

$= (\alpha_k \beta^*_k \pm \alpha_k \beta^*_k) \delta_{\vec{k},-\vec{k}'}$ → These deltas

$= (\alpha_k \beta^*_k \pm \alpha_k \beta^*_k) \delta_{\vec{k},-\vec{k}'}$ ← are over $\vec{k}$ not k Therefore $\delta_{\vec{k},-\vec{k}'}$ means $k = k'$

Morale of the Story:

We see that if β_k is non zero, that is the space is not flat Then the only choice of commutators for the spin 0 field which is consistent with the dynamics is Bose-Einstien and $[\ ]_\pm \rightarrow [\ ]_- \rightarrow [\ ]$ and we find $[a_{\vec{k}}, a_{\vec{k}'}] = 0$; with $1 = |\alpha_k| - |\beta_k|$ (a condition imposed by the field equations)

$$[a_{\vec{k}}, a^+_{\vec{k}'}]_\pm = (|\alpha_k| \pm |\beta_k|)\delta_{\vec{k},\vec{k}'}$$
$$= (1 + |\beta_k| \pm |\beta_k|)\delta_{\vec{k},\vec{k}'}$$

Then since $[\ ]_\pm \longrightarrow [\ ]$

$$[a_{\vec{k}}, a^+_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \checkmark$$

This is truly a curved space relation/connection between spin and statistics. for if $a(t)$ is constant (minkowski) Then $\beta_k = 0$. and this connection vanishes. The curvature of the space is sensitive to spin-statistics.

2.) Probability Distribution of Created Particles

We've had the chance to look at the some what curious (somewhat, in hindsight, obvious if you've studied spinors on a curved space) ability of a spacetime time to use dynamics to enforce proper spin statistics on fields. lets now turn to the probability distributions of these particles we've encountered at late times in our choosen space time.

$$ds^2 = dt^2 - a(t)(dx_i dx^i)$$

as before $\begin{cases} A_{\vec{k}}, A_{\vec{k}}^{+} \text{ - Early times (ET)} \\ a_{\vec{k}}, a_{\vec{k}}^{+} \text{ - late times (LT)} \end{cases}$

and $a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{+}$. $a_{\vec{k}}^{+} = \alpha_k^* A_{\vec{k}}^{+} + \beta_k A_{-\vec{k}}$

We will first see that at LT that particles are created in pairs of opposite momentum states.

let $|0\rangle$ be the vacuum at ET

i.e. $A_{\vec{k}}|0\rangle = 0 \quad \forall \vec{k}$

; let $|0)$ (curved braket) be the vacuum at LT

i.e. $a_{\vec{k}}|0) = 0 \quad \forall \vec{k}$

our late time fock space is created by acting with $a_{\vec{k}}^{+}$ on $|0)$. let us consider the amplitude for finding the state

$|n(k), n(-k))$ where n is the number of particles with momentum k.

Clearly this is $(n(k), n(-k)|0\rangle$

(Recall $a_{\vec{k}}^{+n}|0) = \sqrt{n!}\,|n(k))$)

$\Rightarrow$ $(n(k), n(-k)|0\rangle$

$= (0|\, n!^{-1} (a_{\vec{k}})^n (a_{-\vec{k}})^n |0\rangle \quad *$

now

$a_{\vec{k}}|0\rangle = (\alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^{+})|0\rangle = \beta_k^* A_{-\vec{k}}^{+}|0\rangle$

$= \beta_k^* \alpha_k^{*-1}(a_{-\vec{k}}^{+} - \beta_k A_{\vec{k}})|0\rangle = \frac{\beta_k^*}{\alpha_k^*} a_{-\vec{k}}^{+}|0\rangle$

use this in $*$

$(0|\, n!^{-1} (a_{-k})^n \left(\frac{\beta_k^*}{\alpha_k^*}\right)^n (a_{-k}^{+})^n |0\rangle$

$= n!^{-1/2} \left(\frac{\beta_k^*}{\alpha_k^*}\right)^n (n(-k)|\,(a_{-k}^{+})^n|0\rangle$

$= \left(\frac{\beta_k^*}{\alpha_k^*}\right)^n (0|0\rangle$

Suppose now we want $(m(k), n(-k)|0\rangle$ with $m<n$. Then looking back at the calculation we see that $(m(-k)|(a^+_{-k})^n = 0$

now suppose $m>n$ (I won't prove this in general but to get the idea let $m=n+1$

$$\Rightarrow \quad ((n+1)(-k)|(a^+_{-k})^n|0\rangle$$
$$= (-k|0\rangle$$
$$= (0|a_{-k}|0\rangle$$
$$= (0|\alpha_k A_{-k} + \beta^*_n A^+_{+k}|0\rangle$$
$$= (0|\beta^*_k \alpha^*_k (a^+_k)|0\rangle$$
$$= 0$$

(HW: use induction to show this holds for general n; m. or just find a clever way to show it, I can't see it right now).

[HW = Homework]

Morale of the story:
Particles are created in pairs of equal and opposite momenta. It follows that the most general LT state with non-zero matrix elements with the ET vacuum is a tensor product of states $|\{n_j(k_j)\}) \equiv |n_1(\vec{k}_1), n_1(-\vec{k}_1)) \otimes \cdots \otimes |n_j(\vec{k}_j) n_j(-\vec{k}_j))$

Now lets consider the general amplitude

$(\{n_j(k_j)\}|0\rangle$ using our previous result

$$\boxed{(\{n_j(k_j)\}|0\rangle = \prod_j \left(\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right)^{n_j} (0|0\rangle}$$

Furthermore

Expansion in a complete set of states.

$$\boxed{|0\rangle = \sum_{\{n_j(k_j)\}} |\{n_j(k_j)\})(\{n_j(k_j)\}|0\rangle}$$

The sum is over all sets $\{n_j(k_j)\}$

normalize $\langle 0|0\rangle = 1$

$$1 = \langle 0|0\rangle \stackrel{?}{=} \sum_{\{n_j(k_j)\}}\sum_{\{n_i(k_i)\}} \langle 0|\{n_j(k_j)\}) \underbrace{(\{n_j(k_j)\}|\{n_i(k_i)\})}_{\delta_{\{n_j\},\{n_i\}}} (\{n_i(k_i)\}|0\rangle$$

$$= \sum_{\{n_j(k_j)\}} \langle 0|\{n_j(k_j)\})(\{n_j(k_j)\}|0\rangle$$

$$= \sum_{\{n_j(k_j)\}} |(\{n_j(k_j)\}|0\rangle|^2$$

$$= \sum_{\{n_j(k_j)\}} \left|\prod_j \left(\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right)^{n_j} (0|0\rangle\right|^2$$

$$= \sum_{\{n_j(k_j)\}} \prod_j \left|\left(\frac{\beta^*_{k}}{\alpha^*_{k_j}}\right)\right|^{2n_j} |(0|0\rangle|^2$$

Provided the sums $\sum_i \prod$ converge we can swith the order

$$= \prod_j \underbrace{\sum_{n_j=0}^{\infty} \left|\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right|^{2n_j}}_{\text{Geometric Series}} |(0|0\rangle|^2$$

$$= |(0|0\rangle|^2 \prod_j \left(\frac{1}{1-\left|\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right|^2}\right), \qquad 1 = |\alpha_{k_j}|^2 - |\beta_{k_j}|^2$$

$$1 = |(0|0\rangle|^2 \prod_j |\alpha_{k_j}^*|^{2}$$

finally we obtain $|(0|0\rangle|^2 = \prod_j |\alpha^*_{k_j}|^{-2}$

and the Probability of obtaining the general state $|\{n_j(k_j)\})$ is

$$|(\{n_j(k_j)\}|0\rangle|^2 = \prod_j \left|\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right|^{2n_j} |(0|0\rangle|^2$$

$$= \prod_j \frac{1}{|\alpha_{k_j}|^2}\left|\frac{\beta^*_{k_j}}{\alpha^*_{k_j}}\right|^{2n_j}$$

Morale of the story:
We find that the production of Particles is in Pairs. The Probability of observing n of these bosons in mode k is,

$$P_n = \frac{1}{|\alpha_k|^2}\left|\frac{\beta_k^*}{\alpha_k^*}\right|^{2n}$$

$$\therefore \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty}\left(\left|\frac{\beta_k^*}{\alpha_k^*}\right|^2\right)^n \frac{1}{|\alpha_k|^2}$$

$$= \frac{1}{1-\left|\frac{\beta_k^*}{\alpha_k^*}\right|^2}\frac{1}{|\alpha_k|^2}$$

$$= \frac{1}{\frac{1}{|\alpha_k|^2}}\frac{1}{|\alpha_k|^2} = 1$$

We can then find the expectation value of the number operator at $t\to\infty$ (LT), or the average # of Particles with mode k in a volume $(La_2)^3$ as

$$\langle N_{\vec{k}}\rangle_{LT} = \sum_{n=0}^{\infty} nP_n = \frac{1}{|\alpha_k|^2}\sum_{n=0}^{\infty}\left(\left|\frac{\beta_k}{\alpha_k}\right|^2\right)^n n$$

$$= \frac{1}{|\alpha_k|^2}\sum_{n=1}^{\infty} n\left(\left|\frac{\beta_k}{\alpha_k}\right|^2\right)^n$$

$$= \frac{1}{|\alpha_k|^2}\frac{\left|\frac{\beta_k}{\alpha_k}\right|^2}{\left(1-\left|\frac{\beta_k}{\alpha_k}\right|^2\right)^2} = \frac{|\beta_k|^2}{|\alpha_k|^4}\frac{1}{\left(\frac{1}{|\alpha_k|^2}\right)^2}$$

$$= |\beta_k|^2$$

Then the average Particle density summed over all modes in the continuum limit is

$$\langle N_i\rangle_{t\to\infty} = \lim_{L\to\infty}(La_2)^{-3}\sum_k |\beta_k|^2$$

$$= (2\pi^2 a_2^3)^{-1}\int_0^{\infty} dk\, k^2|\beta_k|^2$$

it turns out there is multiple examples out exactly soluble systems. For the sake of brevity I will refer you to Page 61-63 of Parker & toms. for a detailed discussion. In short

$$ds^2 = dt^2 - a(t)(dx_i dx^i)$$

with $a(\tau) = \left\{ a_1^4 + e^{\xi} \frac{\left[(a_2^4 - a_1^4)(e^{\xi}+1)+b\right]}{(e^{\xi}+1)^2} \right\}^{1/4}$

with $\xi = \tau s^{-1}$

Parameters: $a_1, a_2, b, s > 0$

Behavior: $\tau \to -\infty \quad a(\tau) = a_1$

$\tau \to \infty \quad a(\tau) = a_2$

passes through maximum if $b > |a_2^4 - a_1^4|$

This leads to $\phi_{k(ET)} \sim N_1 u^{-c_1}(1+u)^d F(d-c_1+c_2, d-c_1-c_2, 1-2c_1, -u)$

$\phi_{k(LT)} \sim N_1 A u^{-c_2} + N_1 B u^{c_2}$

Please see Parker and toms for def$^{\underline{n}}$ of relavant quantites.

The bottom line is we can explicitly write $|\beta_k/\alpha_k|$ for this model

$$\left|\frac{\beta_k}{\alpha_k}\right|^2 = \frac{\sin^2\pi d + \sinh^2(\pi k s(a_1^2 - a_2^2))}{\sin^2\pi d + \sinh^2(\pi k s(a_1^2 + a_2^2))}$$

$$d = \tfrac{1}{2}\left(1 - (1 + 4k^2 s^2 b)^{1/2}\right)$$

let us now consider the probability of observing n particles for this model when $k \gg 1$ and $a_1 \neq a_2$. Put $a_{\lessgtr} = \begin{cases} a_< & \min(a_1, a_2) \\ a_> & \max(a_1, a_2) \end{cases}$

for this case sinh is much larger then sin.

$$\epsilon = \left| \frac{\beta_k}{\alpha_k} \right|^2 \cong \frac{\sinh^2(\pi k s(a_1^2 - a_2^2))}{\sinh^2(\pi k s(a_1^2 + a_2^2))}, \quad \sinh x = \frac{e^x - e^{-x}}{2} \cong \frac{e^x}{2} \quad x \gg 1$$

$$\cong \frac{\exp(2\pi k s(a_>^2 - a_<^2))}{\exp(2\pi k s(a_>^2 + a_<^2))}$$

$$\epsilon \cong e^{-\mu k}, \quad \mu = 4\pi s a_<^2$$

now for an expanding universe for which $a_2 = a_>$; $a_1 = a_<$; $a_2 \gg a_1$

ϵ would be an appropriete distribution for all but extremely low frequencies. This is because the low end of the spectrum is effectively Redshifted away.

$$1 = |\alpha_k|^2 - |\beta_k|^2$$

$$\frac{1}{|\alpha_k|^2} = 1 - \left|\frac{\beta_k}{\alpha_k}\right|^2$$

We would find then

$$P_n = \frac{1}{|\alpha_k|^2} \left| \frac{\beta_k}{\alpha_k} \right|^{2n} = \left(1 - \left|\frac{\beta_k}{\alpha_k}\right|^2\right) \left| \frac{\beta_k}{\alpha_k} \right|^{2n}$$

$$P_n = (1 - e^{-\mu k}) e^{-\mu n k}, \quad \mu = 4\pi s a_1^2$$

Then the average # in mode N is

$$|\beta_k|^2 = \frac{|\beta_k|^2}{1} = \frac{|\beta_k|^2}{|\alpha_k|^2 - |\beta_k|^2} = \frac{|\beta_k/\alpha_k|^2}{1 - |\beta_k/\alpha_k|^2}$$

$$= \frac{e^{-\mu k}}{1 - e^{-\mu k}} = \frac{1}{e^{\mu k} - 1}$$

The average Particle density in all modes

$$\langle N \rangle = \frac{1}{(2\pi^2 a^3)} \int_0^\infty dk\, k^2 \frac{1}{e^{\mu k} - 1}$$

We see this is an aproximation of the blackbody radition (here scalar radiation).

Morale of the Story:

The late time vacuum is a coherent superposition of states containing Pairs of Particles. This different to blackbody radiation (incoherent mixture).
However local observations would be unable this radiation and blackbody radiation since correlated Pairs of Particles would have cosmological separations.
Hence we would observe "locally" the creation of a thermal bath of Particles in an expanding universe.