LINEAR Vs. NON-LINEAR PDE & SUPERPOSITION PRINCIPLE

Consider a general PDE as follows
\[ U^{(\alpha_1, \ldots, \alpha_n)}(x_1, \ldots, x_n) \sum_{i=1}^{n} \frac{\partial^{\alpha_i} U}{\partial x_i^{\alpha_i}} = 0, \]

where \( x_i \)'s are independent variables.

\* Linear PDE \( (\text{Diff p} = 0) \)

- Linear PDE obeys Superposition principle
  - If \( u_1 \) & \( u_2 \) are solution to the PDE then \( (a_1 u_1 + a_2 u_2) \) is also solution to the PDE.

1. \( \frac{\partial}{\partial x} \left( a_1 u_1 + a_2 u_2 \right) = a_1 \frac{\partial u_1}{\partial x} + a_2 \frac{\partial u_2}{\partial x} = 0 \)

\[ \Rightarrow \frac{\partial}{\partial x} (a_1 u_1 + a_2 u_2) = 0 \quad \text{QED} \]

Proof:

Let \( \Delta = \partial^2_x + \partial^2_y \) (second order PDE).

- \( \alpha = 1, \beta = 0 \)
  \[ \frac{\partial}{\partial x} (a_1 u_1 + a_2 u_2) = a_1 \frac{\partial u_1}{\partial x} + a_2 \frac{\partial u_2}{\partial x} = 0 \]

- \( \alpha = 2, \beta = 0 \)
  \[ \frac{\partial^2}{\partial x^2} (a_1 u_1 + a_2 u_2)^2 = 2 a_1 a_2 \frac{\partial^2 u_1 u_2}{\partial x^2} + 2 a_1 a_2 \frac{\partial^2 u_1 u_2}{\partial x \partial y} + 2 a_1 a_2 \frac{\partial^2 u_1 u_2}{\partial y^2} = 0 \]

- \( \alpha = 1, \beta = 1 \)
  \[ (a_1 u_1 + a_2 u_2) \partial^2 (a_1 u_1 + a_2 u_2) = a_1^2 \partial^2 u_1 + a_2^2 \partial^2 u_2 + a_1 a_2 \partial^2 (u_1 u_2) + a_1 \partial^2 (u_1 u_2) + a_2 \partial^2 (u_1 u_2) \]

Note: Only \( U_1 \partial^2 U_1 = 0 \) from original equation.
**General Solution to Linear PDE**

For now we focus on homogeneous Linear Equations i.e No Source/driving term in PDE

\[ \Delta [u] = g(x_1, \ldots, x_n) \quad \text{and} \quad g(x_1, \ldots, x_n) = 0 \]

then using superposition principle we have:

\[ \hat{U} = \sum \hat{c}_i \hat{u}_i(x_1, \ldots, x_n) \]

where \( c_i \)'s are constants fixed by boundary conditions.

**Example**

1. Schrödinger Equation:
   
   - Individual SSF (Eigenfunctions) \( \Rightarrow \) "Non-Localized"
   - Superposition \( \Rightarrow \) Gaussian Wavepacket (Localized in time & space) = "Particle"
   - But is the localization permanent in time & space
     \( \hat{U}(x, t) \Rightarrow \) Is not localized forever?

   We have dispersion because
   
   (Phase-velocity) \( V_p = f(k) \) where \( k \) = wavelength
   
   \[ \text{equivalently} \quad \omega = \omega(k) \]
   
   \[ \text{i.e.} \quad V_g = \frac{d\omega}{dk} \]

   Such that; \( V_g \neq V_p \Rightarrow \text{Dispersion Occurs} \)

   **Free Particle:** \( (\epsilon = \frac{1}{2}) \)
   
   \[ \frac{1}{2}mv_p^2 = \frac{1}{2} \implies V_p = \frac{p}{m} \quad \Rightarrow \quad \left[ V_p = \frac{\hbar k}{m} \right] \]

   \[ E = \hbar \omega = \frac{p^2}{2m} \implies \omega = \frac{\hbar k^2}{2m} \quad \Rightarrow V_g = \frac{\hbar k}{2m} \quad \Rightarrow \quad \left[ V_g = \frac{1}{2} V_p \right] \]

   \[ \Rightarrow \text{Dispersion} \]
Relativistic Free particle

⇒ Dirac Equation (Not Correct treatment ⇔ Negative Probabilities)

Solution:

"Quantum Field Theory"

Conclusion from general solution of Linear PDE

- There is always dispersion!!
- It manifests as the spread in energy density function

\[ E(x) \rightarrow E(x, t_0) = 0. \]

(localized in x)

Non-linear Equation

In general, like L-PDE, are solutions to NL-PDE "dissipative"?

1877: Boussinesq

1895: Diederik Korteweg & Gustav de Vries (Rediscovered)

\[ \partial_t \phi + \partial_x^3 \phi + 6 \phi \partial_x \phi = 0 \]

If \( \phi(x, t) = f(x - c t + a) \equiv f(X) \)

\[ \Rightarrow -c \partial_X f + \partial_X^3 f + 6 f \partial_X f = 0 \]

Integrating with \( X \):

\[ \Rightarrow -c f + \partial_X^2 f + 3 f^2 = A \text{ (int constant)} \]
\[-c f + \frac{d^2}{dx^2} f + 3f^2 = A\]

\begin{itemize}
  \item Newton's Eq. for Cubic Potential
    \begin{equation*}
      \frac{d^2}{dx^2} f = -\nabla V(x)
    \end{equation*}
  \end{itemize}

Solution:
- Fix \( V \) such that \( V(f=0) \) yields local maxima.
- Then,
  \[ \phi(x,t) = \frac{1}{2} c \text{ Sech} \left[ \frac{\sqrt{c}}{2} (x - ct - a) \right] \]

\begin{itemize}
  \item Interaction of two solitary waves: (One has larger amplitude)
    \begin{itemize}
      \item Non-trivial
      \item Closer analysis shows that when two of these solitary waves meet larger wave transfers energy to smaller wave such that it appears as if they pass right through each other
    \end{itemize}
\end{itemize}

[Mathematically]
\[ V > 2 \]
Integral of Miscon

- Integral of momentum define the solutions that do not evolve with time.
- KdV integral of momentum include
  - mass
  - momentum
  - energy

Field Theory & Particles: "Aka Solitons"

- Particles are defined as localized "lumps" of energy that propagate.
- Lumps: Coined by Sidney Coleman to distinguish between solitons which have more precise & narrow definition in mathematics.
- Solitary waves $\Rightarrow$ Solitons ("ons" is suffix to imply particle-like).
- Solitons propagate without dissipation.
- Example of Soliton Like Solutions
  - Euclidean Yang-Mills Equation.
  - Finite energy solution have important interpretation in gauge theory.

$\Rightarrow$ "Instantons": Localized in imaginary time.

\[ (\text{Alexandre Polyakov did this for classical Yang-Mills Eq}) \quad (1970) \]

$\Rightarrow$ 't Hooft used the quantum version to study the ground state of the model. (Famously solving so-called UV/IR problem)

$\Rightarrow$ Attempts to understand vacuum of higher dim gauge theory.
\[ L = T - V = \frac{1}{2} (\partial \phi)^2 - V(\phi) \]

**EOM**

\[
\frac{\partial}{\partial x} \left( \frac{\partial L}{\partial (\partial \phi)} \right) - \frac{\partial L}{\partial \phi} = 0
\]

\[ \Rightarrow \Box \phi(x,t) + V'(\phi) = 0 \]

(Non-Linear Part)

> **Finite Energy**

\[ \int_{|x|<\infty} |x|^2 (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + V(\phi)^2 \, dx < \infty \]

> **Convergence**

- Each integrand is positive definite \( \forall x \), convergence requires

\[ \{ \partial_t \phi, \partial_x \phi, V(\phi) \} \xrightarrow{\infty} 0 \]

i.e. \( \phi(x,t) \rightarrow \text{constant, for } |x| \rightarrow \text{large} \)

Then, \[ H(\phi = a) = 0 \] , here \( \phi(x) = a \) is the classical ground state of the system

> Also, (momentum)

\[ T_{01}(x,t) = \tilde{P}(x,t) = (\partial_t \phi) (\partial_x \phi) \iff \text{Momentum Density} \]

Total Momentum \( P(x,t) = \int_{-\infty}^{\infty} \tilde{P}(x,t) \, dx \)
Examples:

a) K-G Equation \( V(\phi) = m^2 \phi^2 \)
   
   Ground state: \( \phi_g = 0 \) \( \Rightarrow \) Trivial solution.

b) \( \phi^4 \) theory:
   
   \[ V = \lambda (\phi^2 - v^2) \]
   
   \( \Rightarrow \phi_g = \pm \sqrt{V} \) \( \Rightarrow \) 2 vacuums

c) Sine-Gordon Equation
   
   \[ V = \alpha (1 - \cos \beta \phi) \; (\alpha, \beta > 0) \]
   
   \[ \Rightarrow \phi_g = \frac{2\pi n}{\beta} \] \( \Rightarrow \) Multiple vacuums

d) Liouville Equation
   
   \[ V(\phi) = e^{\lambda \phi} \]
   
   No, classical ground state

Aside:

In QFT \( \langle \phi_g \rangle = \text{Vacuum} \)

And,

We expand the solution around this classical vacuum & quantize the fluctuation which we interpret as

\[ \text{Particle} \sim \text{quant of the field} \]

\( \Rightarrow \) Fluctuation of field expands around classical vacuum of the theory
General Analysis

We want to find time-independent solution to our EOM

\[ \Phi(x,t) \equiv \Phi(x) \]

\[
\text{EOM} \quad -\frac{d^2 \Phi}{dt^2} + V(\Phi) = 0
\]

where \[ V' \equiv \frac{dV}{d\Phi} \]

Combine with Newton's Equation: \[ \frac{d^2 x}{dt^2} = F = -\frac{dV}{dx} \]

So the solution to our potential is found by identification that

\[ X \leftrightarrow \Phi \quad \& \quad V_x \leftrightarrow -V(\Phi) \]

ie It is same as particle with unit mass moving in a inverted potential

We can write a Lagrangian to give the same EOM as

\[ L = \frac{1}{2} \left( \frac{d\Phi}{dx} \right)^2 - V(\Phi) \quad \leftrightarrow \quad T - V \]

We can then directly write the energy density for this;

\[ \mathcal{N} = \frac{1}{2} \left( \frac{d\Phi}{dx} \right)^2 - V(\Phi) \]

\[ \frac{d\mathcal{N}}{dt} \]

\[ \frac{d\mathcal{N}}{dx} = 0 \]

\[ \text{Position Independent energy density} \]

\[ \text{General Constraint} \]

\[ \text{Finite total energy constraint requires convergence of integrand of } \mathcal{N} \]

ie \[ \{(\Phi(x), V(\Phi)) \} \xrightarrow{(x \to \pm \infty)} \emptyset \]
So we have;

\[ H(x \rightarrow \pm \infty) = 0 \]  

Clearly this is consistent to requirement \( \frac{dH}{dx} = 0 \)

So we conclude that;

Classical Ground State is indeed the solution!!

\[ \text{★ Equi-partition} \]

We have, \( H = 0 \) \( \Rightarrow \) \( \frac{1}{2} (\varphi(x))^2 = V(\varphi) \)

\[ \Rightarrow \quad \varphi'(x) = \pm \sqrt{2V(\varphi)} \]

So we have two \textbf{Solution} solutions: \( \{+\} \iff \{\text{Solution, anti-solution}\} \)

\[ \text{★ Ground States (classical) } \Rightarrow \quad |H| = 0 \]

Analogously we have two solutions for ground state at two asymptotes

at \( |x \rightarrow \pm \infty| \) such that

\[ \{ \varphi', V(\varphi) \} \mid_{x \rightarrow \pm \infty} = \emptyset \]

\[ \in \quad |H| \mid_{\pm} = 0 \]

we identify these asymptotes \( (x \rightarrow \pm \infty) \) happens for sufficiently large \( x \)
Now, \( \phi' (\pm \infty) = 0 \Rightarrow \phi (\pm \infty) = \text{constant} = \pm \theta \)

\( V (\pm \theta) = 0 \)

So, we have two minima & time-independent \( \phi (t) \) is thought as solution that tunnels from one vacuum to the other.

\[ \Gamma^{\mu \nu} = \frac{2L}{2 \partial^2 \phi} \left( \frac{\partial \phi}{\partial x^\mu} \right) - \frac{8 \pi L^2}{2} \]

\( \hat{H} = 2V (\pm \theta) = \widehat{T} \)

\( H = \int_{-\infty}^{\infty} (\partial_x \phi)^2 \, dx = 2 \int_{-\infty}^{\infty} V (\pm \theta) \, dx \)

\[ H_{\pm} = \int_{-\infty}^{\infty} (\partial_x \phi)^2 \, dx = 0 \]

\[ P_{\pm} = \int (\partial_t \phi) \partial_t (\pm \theta) \, dx = 0 \]

\textbf{Time-dependent Solution}

- \( \phi_{\pm} \) are time-independent solution at \( t = \pm \infty \)
- For other times \( t \), we use Lorentz boost:

\[ x \rightarrow Y (x + \beta t) \quad \text{;} \quad Y = \frac{1}{1 - \gamma^2} \]
So, time-dependent solution are given by:

\[ \varphi_{\pm, \beta}(x, t) = \varphi_{\pm}(x_{\beta}) \quad ; \quad x_{\beta} = X(x + \beta t) \]

One can show that this is a solution to the original time-dependent equation

\[ \square \varphi_{\pm, \beta}(x_{\beta}) = -V(\varphi_{\pm, \beta}) \]

Similarly,

\[
\begin{pmatrix}
H_{\pm, \beta} \\
P_{\pm, \beta}
\end{pmatrix}
= \begin{pmatrix}
\cosh \lambda & -\sinh \lambda \\
\sinh \lambda & \cosh \lambda
\end{pmatrix}
\begin{pmatrix}
H_{\pm, 0} \\
P_{\pm, 0}
\end{pmatrix}
\]

Where,

1. \(\cosh \lambda = \frac{1}{\sqrt{1 - \nu^2}} \equiv \gamma\)
2. \(\tanh \lambda = \beta = \sqrt{\nu}\)
3. \(\sinh \lambda = \sqrt{\gamma}\)

\[
\Rightarrow \begin{pmatrix}
H_{\pm, \beta} \\
P_{\pm, \beta}
\end{pmatrix}
= \begin{pmatrix}
\gamma & -\sqrt{\nu} \\
+\nu \sqrt{\gamma} & \gamma
\end{pmatrix}
\begin{pmatrix}
H_{\pm, 0} \\
P_{\pm, 0}
\end{pmatrix}
\]
Examples

(a) $V = \frac{1}{2} (\phi^2 - 1)^2$

- $H = 0 \Rightarrow \phi_\pm(x) = \pm 1$
- There are classical ground state of completely stationary fields
  \[ \phi_g(x,t) = \pm 1 \]

- Stationary Soliton
  - $\phi_{\pm, a}(x) = \pm \tanh(x-a)$
  - $\phi_{\pm}(x,t) = \pm \tanh(x-a)$

\[ \phi(x,t) \]

x = +\infty (S\text{olitons})

x = -\infty (antis\text{olitons})

- $H_{\pm, a}(x, t) = \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right]_{x = \pm\infty} = \frac{f}{\cosh^4(x-a)}$

Note that this has no time dependence
\[ H_{\pm,a} = 0 \]

So,

\[ H_{\pm,a} = \int_{-\infty}^{\infty} H_{\pm,a}(x,t) \, dx = \frac{4}{3} \]

\[ P_{\pm,a} = \int_{-\infty}^{\infty} P_{\pm,a}(x,t) \, dx = 0 \]

\[ \text{General Analysis} \]

\[ V = \frac{1}{2} \left( q^2 - \frac{m^2}{2} \right)^2 \]

- \[ \phi_{\pm,q} = \pm \frac{m}{2} \]

- \[ V''(\phi_{\pm,q}) = (2m)^2 \text{ (Curvature)} \]

- Classical length scale \[ \xi = \frac{1}{\sqrt{V''(0)}} = \frac{1}{2m} \]
\[ \phi_{\pm} = \pm \frac{1}{2} \left( m^2 \lambda - C_{\pm}^2 \right) \]

\[ \implies \phi_{\pm, a}(x) = \pm \frac{m}{2} \tanh m(x-a) \]

\[ \implies \phi_{\pm, a}(x+r) = \pm \frac{m}{2} \tanh m(x-a) \]

\[ \mathcal{H}_{\pm, a}(x+r) = \frac{m^4}{2 \cosh m(x-a)} \]

\[ H_{\pm, a}(x+r) = \frac{4m^3}{3 \lambda} \]

**Note:** \( a \) is a free parameter that fixes the center about which energy density is concentrated.
- Tunneling
  \[ \text{eg. from } \phi_{+a}(\infty, t) \text{ to } \phi_{-a}(\infty, t) \]
  \[ \text{Energy barrier } \ H = \frac{\hbar m^3}{\ell} \]
  This suppresses the tunneling rate.
  \[ \text{Q.M. Probability } \sim e^{-2H} \]

- Boosted
  \[ \phi_{\pm, a, \beta}(x, t) = \pm \tan \left( \frac{x - a - \beta t}{\sqrt{1 - \beta^2}} \right) \]

- \( \mathcal{H}(\phi_{\pm, a, \beta}) = \cosh \frac{4m^3}{\mathcal{A}} = \sqrt{(\frac{4m^3}{\mathcal{A}})} \)
- \( P_{\pm, a, \beta} = \sinh \frac{4m^3}{\mathcal{A}} = (\mathcal{V}) \left( \frac{4m^3}{\mathcal{A}} \right) \)
\[ M = \frac{4m^3}{3a} \text{ (rest mass)} \]

\[ \text{Half width} = \frac{\sqrt{1-\beta^2}}{4m} \]

- **Note**
  - Boosted soliton still have no dispersion
  - The shape is altered by boost.

- Sine Gordon Equation

  - \( \phi_{\text{am}} (x,t) = 2ax \) (Degenerate Vacua)

  - \( \phi_{\pm a} (x) = 4 \arctan e^{\pm (x-a)} \)