LINEAR VS. NON-LINEAR PDE & SUPERPOSITION PRINCIPLE

Consider a general PDE as follows

$$U^{\beta}(x_1 \dots x_N)\left\{\hat{\mathcal{L}}(x_1 \dots x_N)\, U^{\alpha}(x_1 \dots x_N)\right\} = 0,$$

where x_i's are independent variables.

☆ <u>Linear PDE</u> (Iff $\beta = 0$)

- Linear PDE obeys Superposition principle

ie If u_1 & u_2 are solution to the PDE then $(a_1 u_1 + a_2 u_2)$ is also solution to the PDE.

i.e. $\left.\begin{array}{l} \text{If } \mathcal{L}(u_1) = 0 \\ \mathcal{L}(u_2) = 0 \end{array}\right\} \mathcal{L}(a_1 u_1 + a_2 u_2) = a_1 \underbrace{\mathcal{L}(u_1)}_{0} + a_2 \underbrace{\mathcal{L}(u_2)}_{0} = 0$

So, $\boxed{\mathcal{L}(a_1 u_1 + a_2 u_2) = 0}$ <u>QED</u>

<u>Proof:</u>

Let $\hat{\mathcal{L}} = \partial_x^2 + \partial_y^2$ (second order PDE).

- $\alpha = 1,\ \beta = 0$

$$\mathcal{L}(a_1 u_1 + a_2 u_2) = a_1 \underbrace{\mathcal{L}(u_1)}_{0} + a_2 \underbrace{\mathcal{L}(u_2)}_{0} = \boxed{0}$$

- $\alpha = 2\ ;\ \beta = 0$

$$\mathcal{L}(a_1 u_1 + a_2 u_2)^2 = \mathcal{L}(a_1 u_1 + a_2 u_2)^2 = \hat{\mathcal{L}}(a_1 u_1)^2 + \hat{\mathcal{L}}(a_2 u_2)^2 + \underbrace{2\hat{\mathcal{L}}(a_1 a_2 u_1 u_2)}$$

$$= 2a_1 u_1 \cancelto{0}{\mathcal{L}(u_1)} + 2a_2 u_2 \cancelto{0}{\mathcal{L}(u_2)} + 2a_1 a_2 (u_1 \cancelto{0}{\mathcal{L}(u_2)} + \cancelto{0}{\mathcal{L}(u_1)}\, u_2)$$

$$= \boxed{0}$$

- $\alpha = 1\ ;\ \beta = 1$

$$(a_1 u_1 + a_2 u_2)\, \hat{\mathcal{L}}(a_1 u_1 + a_2 u_2) = a_1^2 \underbrace{u_1 \hat{\mathcal{L}}(u_1)}_{\cancel{0}} + a_2^2 \underbrace{u_2 \hat{\mathcal{L}}(u_2)}_{\cancel{0}} + a_1 a_2 \left[\underbrace{u_1 \hat{\mathcal{L}}(u_2)}_{??} + \underbrace{u_2 \hat{\mathcal{L}}(u_1)}_{??}\right]$$

<u>Note</u>: Only $\boxed{\begin{array}{l} u_1 \hat{\mathcal{L}} u_1 = 0 \\ u_1 \mathcal{L} u_2 \neq 0 \end{array}}$ From original equation!!

☆ General Solution to Linear PDE

For now we focus on homogeneous Linear Equations i.e No Source/driving term in PDE

i.e $\hat{\mathcal{L}}[u^{\alpha}] = g(x_1 \ldots x_N)$ & $\boxed{g(x_1 \cdots x_N) \equiv 0}$

then using superposition principle we have:

$$\boxed{\tilde{U} = \sum_i c_i u_i(x_1 \ldots x_N)}$$, where c_i's are constants fixed by boundary cond^n's.

Example

(A) Schrödinger Equation:

- Individual Sol^n (Eigenfunction) $\Rightarrow$ "Non-Localized"
- Superposition $\Rightarrow$ Gaussian Wave packet (Localized in time & space) $\equiv$ "Particle"
- But is the localization permanent in time & space

$\tilde{U}(x,t) \Longrightarrow$ Is not localized forver?

We have dispersion because

(phase-velocity) $V_p = f(k)$. where $k \propto$ wavelength

equivalently $\Leftrightarrow$ $\omega \equiv \omega(k)$

i.e $V_g = \dfrac{d\omega}{dk}$

Such that; $\boxed{V_g \neq V_p} \Longleftrightarrow \boxed{\text{dispersion Occurs}}$

Free particle: $(p = \hbar k)$

- $\frac{1}{2} m v_p^2 = \frac{p^2}{2m} \Rightarrow V_p = \frac{p}{m} \Longrightarrow \boxed{V_p = \frac{\hbar k}{m}}$
- $E = \hbar\omega = \frac{p^2}{2m} \Rightarrow \omega = \frac{\hbar k^2}{2m} \Rightarrow V_g = \frac{d\omega}{dk} = \frac{\hbar k}{2m} \Rightarrow \boxed{V_g = \frac{1}{2} V_p}$

$\Rightarrow$ $\boxed{\text{Dispersion}}$ ✓

☆ Relativistic Free particle

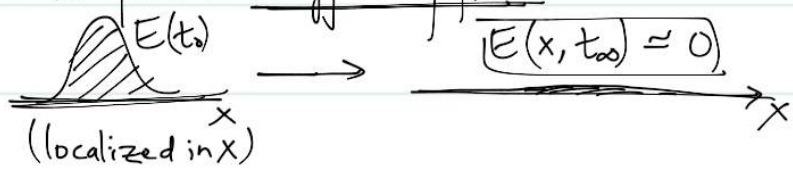
$\Rightarrow$ Dirac Equation (Not Correct treatment $\Leftrightarrow$ "Negative Probabilities"

<u>Solution</u>:

"Quantum Field Theory".

▸ <u>Conclusion from general solution of Linear PDE</u>

- There is always <u>dispersion</u>!!
- It manifests as the spread in <u>energy density function</u>



☆ <u>NON-LINEAR EQUATION</u>

Q In general, like L-PDE, are solutions to NL-PDE "dissapative"?

<u>1877</u>: Boussinesq

<u>1895</u>: Diederik Korteweg & Gustav de Vries (Rediscovered)

} (KdV Equations)

$$\boxed{\partial_t \phi + \partial_x^3 \phi + 6\phi \partial_x \phi = 0}$$

If $\phi(x,t) = f(x - ct + a) \equiv f(X)$

$$\Rightarrow -c\,\partial_X f + \partial_X^3 f + 6f\,\partial_X f = 0$$

Integrating with X;

$$\Rightarrow -cf + \partial_X^2 f + 3f^2 = A \text{ (Int Constant)}$$

$$\boxed{-cf + \partial_x^2 f + 3f^2 = A}$$

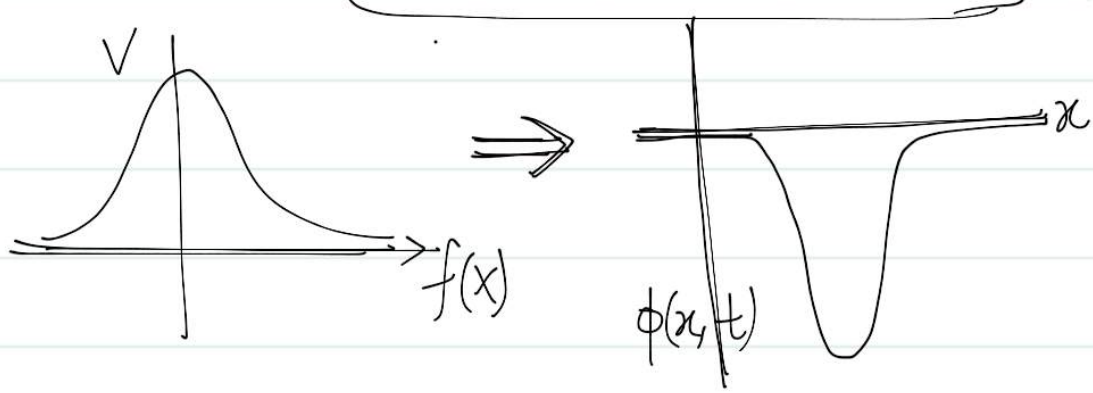
~ Newton's Eq^n for Cubic Potential

ie $\boxed{\partial_x^2 f = -\nabla V(x)}$

<u>Solution</u>:

- Fix V such that $V(f=0) \ni \begin{pmatrix}\text{Local}\\ \text{Maxima}\end{pmatrix}$
- Then,

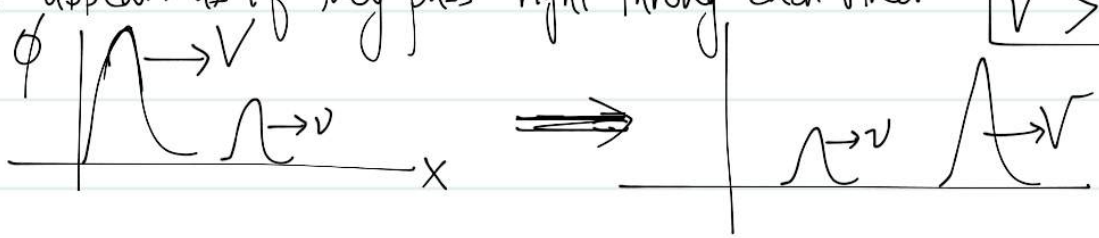
$$\boxed{\phi(x,t) = \frac{1}{2} c \, \mathrm{Sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - a)\right]} \quad \begin{pmatrix}\text{Solitary}\\ \text{Waves}\end{pmatrix}$$



$\Rightarrow$ Interaction Of two <u>Solitary Waves</u>: $\begin{pmatrix}\text{One has larger Amplitude}\\ \text{Compared to the other}\end{pmatrix}$

- Non-trivial
- Closer analysis shows that when two of these solitary waves meet larger wave transfers energy to smaller wave such that it appears as if they pass right throng each other

$\boxed{\begin{matrix}\text{Mathematically}\\ V > v\end{matrix}}$



>> Integral of Motion

- Integral of motion define the solutions that do not evolve with time.
- KdV Integral of motion include
  - mass
  - momentum
  - energy

☆ Field Theory & Particles "Aka Solitons"

- Particles are defined as localized lumps of energy that propagate
- Lumps : Coined by Sydney Coleman to distinguish between solitons which have more precise & narrow definition in mathematics
- Solitary waves ⟹ Solitons ("ons" is suffix to imply particle like)
- Solitons propagate without dissapation
- Example of Soliton Like Solutions
  - Eculidean Yang-Mills Equation
  - Finite energy solution have important interpretation in gauge theory

⟹ "Instantons" : Localized in imaginary time.

(Alexendre Polyakov did this for classical Yang Mills Eqn (1970))

⟹ T-Hooft used the quantum version to study the ground state of the model. (Famously solving so called U(1) problem)

⟹ Attempts to understand vaccum of higher dim gauge theory

☆

$$\mathcal{L} = T - V = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

<u>EOM</u> $\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \left(\frac{\partial V}{\partial \phi}\right) = 0}$

$$\Rightarrow \boxed{\Box\, \phi(x,t) + \underbrace{V'(\phi)}_{\text{(Non-Linear part)}} = 0}$$

> <u>Finite Energy</u> $\Rightarrow$ $\boxed{!(x,t) < \infty}$

$$(x,t) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\underbrace{(\partial_t \phi)^2}_{<\infty} + \frac{1}{2}\underbrace{(\partial_x \phi)^2}_{<\infty} + \underbrace{V(\phi)}_{<\infty} \right\} dx < \infty$$

> <u>Convergence</u>

- Each integrand is positive definite so, Convergence requires

$$\{\partial_t \phi, \partial_x \phi, V(\phi)\} \xrightarrow[(x \to \infty)]{} 0$$

ie $\phi(x,t) \longrightarrow$ constant for ($|x| \longrightarrow$ large)

Then, $\boxed{H(\phi = a) = 0}$, where $\boxed{\phi(x) = a}$ is the classical ground state of the system

> Also, (momentum)

- $T_{01}(x,t) = \tilde{P}(x,t) = (\partial_t \phi)(\partial_x \phi) \Longleftrightarrow$ Momentum Density

$$\text{Total Momentum } (P(x,t)) = \int_{-\infty}^{\infty} \tilde{P}(x,t)\, dx$$

> Examples

a) K-G Equation $(V(\phi) = m^2\phi^2)$

Ground state : $\boxed{\phi_g = 0}$ $\Longrightarrow$ $\boxed{\text{trivial solution}}$

b) ϕ^4 theory :

$$V = \lambda(\phi^2 - v^2)$$

$$\Rightarrow \boxed{\phi_g = \pm\sqrt{v}} \quad (2 \text{ vaccum})$$

c) Sine-Gordon Equation

$$V = \alpha(1 - \cos\beta\phi) \; ; \; (\alpha, \beta) > 0$$

$$\Rightarrow \boxed{\phi_g = 2\pi n/\beta} \quad (\text{multiple vaccum})$$

d) Liouville Equation

$$\boxed{V(\phi) = e^{\lambda\phi}}$$

No, classical ground state

Aside :

In QFT $\boxed{\langle\phi_g\rangle = \text{Vaccum}}$

And,

We expand the solution around this classical vaccum & quantize the fluctuation which we interpret as

$\boxed{\text{particle}}$ ~ "quant of the field".

$\Rightarrow$ Fluctuation of field expanded around classical vaccum of the theory

☆ General Analysis

We want to find time-independent solution to our EOM

So, $\boxed{\phi(x,t) \equiv \varphi(x)}$;

EOM $\boxed{-\frac{d^2\varphi}{dt^2} + V'(\varphi) = 0}$ where $\boxed{V' \equiv \frac{dV}{d\varphi}}$

Combare with Newtons Equation : $\boxed{\frac{d^2x}{dt^2} = F = -\frac{dV_x}{dx}}$ with $t \leftrightarrow x$

So the solution to our potential is found by identification that

$\boxed{X \leftrightarrow \varphi}$ & $\boxed{V_x \leftrightarrow -V(\varphi)}$

ie It is same as particle with unit mass moving in a inverted potential

We can write a Lagrangian to give the same EOM as

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - V(\varphi)} \Leftrightarrow \boxed{T - V}$$

We can then directly write the energy density for this ;

$$\boxed{\mathcal{H} = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - V(\varphi)}$$

» Conservation Of Energy Constraint:

ie $\boxed{\frac{d\mathcal{H}_N}{dt} = 0} \Leftrightarrow \boxed{\frac{d\mathcal{H}}{dx} = 0}$ (Position Independent energy density) (General Constraint)

⊕ Finite total energy constraint requires convergence of integrand of H

i.e $\{\varphi'(x), V(\varphi)\} \xrightarrow[(x \to \pm\infty)]{} \emptyset$

So we have;

$$\boxed{H(x \to \pm\infty) = 0}$$ Clearly this is consistent to requirement $\boxed{dH/dx = 0}$

So we conclude that;

Classical Ground State is indeed the solution!!

» Equipartition

We have, $\boxed{H = 0} \implies \frac{1}{2}(\varphi'(x))^2 = V(\varphi)$

$$\implies \boxed{\varphi'(x) = \pm\sqrt{2V(\varphi)}}$$

So we have two soliton solutions: $\left\{ \begin{matrix} + \\ - \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \text{soliton} \\ \text{anti-soliton} \end{matrix} \right\}$

» Ground States (classical) $\implies \boxed{H = 0}$

Analogously we have two solutions for ground state at two asymptotes at $\boxed{x \to \pm\infty}$ such that

$$\boxed{\{\varphi', V(\varphi)\}\Big|_{x \to \pm\infty} = \emptyset} \quad \& \quad \boxed{H\Big|_{\pm} = 0}$$

we identify these asymptotes $(x \to \pm\infty)$ happens for sufficiently large x

Now, $\begin{cases} \circ\ \phi'(\pm\infty) = 0 \Rightarrow \phi(\pm\infty) = \text{constant} \equiv \pm a \\ \circ\ V(\pm a) = 0 \end{cases}$

- So we have two minima & time-independent $\varphi(x)$ is thought as solution that tunnels from one vaccum to the other.

>> <u>Energy density</u>: $\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial^\nu \phi) - \delta^{\mu\nu}\mathcal{L}}$

Using equipartition theorem we have;

$$\circ\ \hat{\mathcal{H}}_\pm = 2V(\varphi_\pm) = T_{00}$$

$$\circ\ T_{00} \quad \circ\ H_\pm = \int_{-\infty}^{\infty} (\partial_x \varphi)^2\, dx = \boxed{2\int_{-\infty}^{\infty} V(\varphi_\pm)\, dx}$$

>> <u>Momentum</u>

$$\circ\ P_\pm = \int (\partial_t \varphi_\pm)(\partial_t \varphi_\pm)\, dx = \boxed{0}$$

>> <u>Time dependent solution</u>

- $\varphi_\pm$ are time independent solution at $t = \pm\infty$
- For other times in between we use Lorentz boost.

$$\boxed{x \longrightarrow \gamma(x + \beta t)} \ ; \ \boxed{\gamma \equiv \frac{1}{1 - v^2}}$$

-

So, time-dependent solution are given by ;

$$\boxed{\phi_{\pm,\beta}(x,t) = \varphi_{\pm}(x_\beta)} \; ; \; \boxed{x_\beta = \gamma(x + \beta t)}$$

One can show that this is solution to the original time dependent equation

ie $$\boxed{\Box\, \varphi_{\pm,\beta}(x_\beta) = -V(\varphi_{\pm,\beta})}$$

Similarly ;

$$\begin{pmatrix} H_{\pm,\beta} \\ P_{\pm,\beta} \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} H_{\pm,0} \\ P_{\pm,0} \end{pmatrix}$$

Where,

- $\cosh\psi = \dfrac{1}{\sqrt{1-v^2}} \equiv \gamma$
- $\tanh\psi = \beta = v$
- $\sinh\psi = v\gamma$

$$\Rightarrow \boxed{\begin{pmatrix} H_{\pm,\beta} \\ P_{\pm,\beta} \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ +v\gamma & \gamma \end{pmatrix} \begin{pmatrix} H_{\pm,0} \\ P_{\pm,0} \end{pmatrix}}$$

☆ Examples

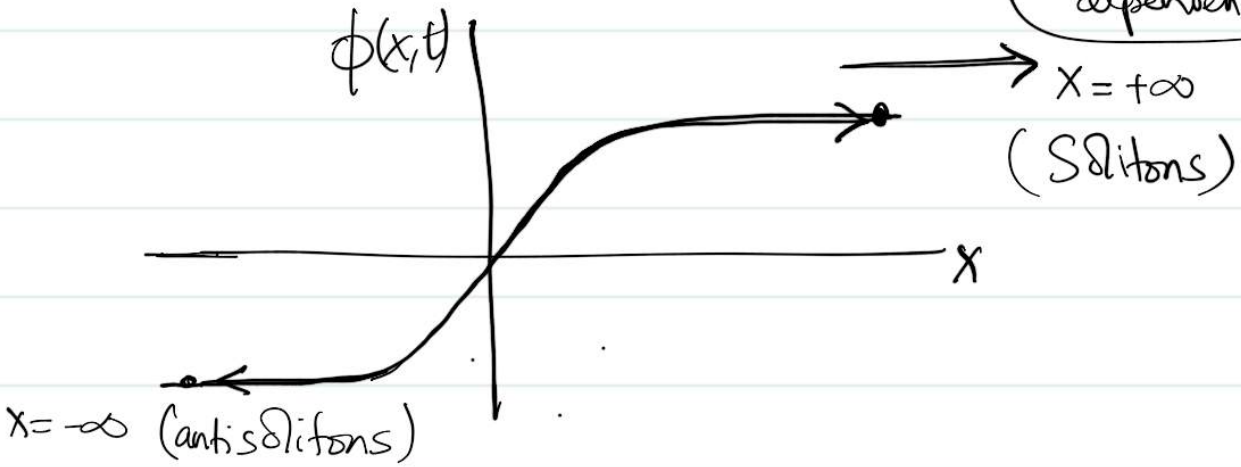
ⓐ $V = \frac{1}{2}(\varphi^2 - 1)^2$

- $\mathcal{H} = 0 \Rightarrow \boxed{\varphi_{g\pm}(x) = \pm 1}$
- These are $\underbrace{\text{classical ground state}}_{\boxed{E=0}}$ of completely stationary fields

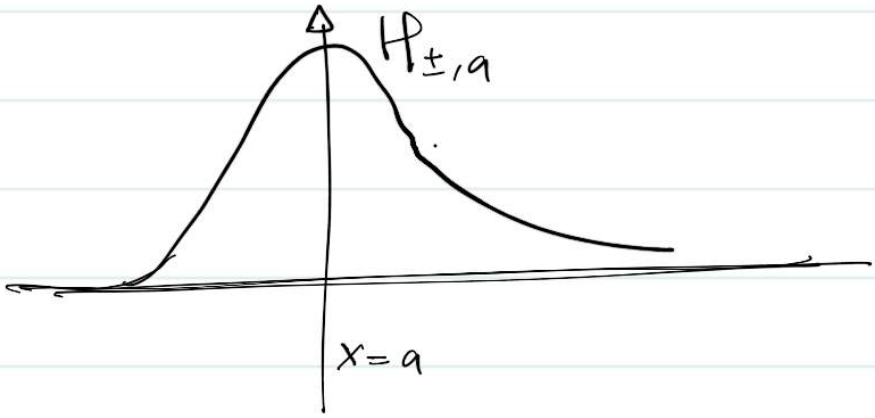
$$\Rightarrow \boxed{\phi_g(x,t) = \pm 1}$$

- Stationary Soliton
  - $\varphi_{\pm,a}(x) = \pm \tanh(x-a)$

  & $\phi_\pm(x,t) = \pm \tanh(x-a)$ Note that this has no time dependence



- $\mathcal{H}_{\pm,a}(x,t) = \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \cancel{V(\phi)}^{\,0}\right]\Big|_{x=\pm\infty} = \boxed{\frac{1}{\cosh^4(x-a)}}$



- $\tilde{P}_{\pm,a}(x,t) = 0$

So,

- $H_{\pm,a} = \int_{-\infty}^{\infty} \mathcal{H}_{\pm,a}(x,t)\, dx = \frac{4}{3}$
- $P_{\pm,a} = \int_{-\infty}^{\infty} P_{\pm,a}(x,t)\, dx = \boxed{0}$

General Analysis

$$V = \frac{\lambda}{2}\left(\varphi^2 - \frac{m^2}{\lambda}\right)^2$$

- $\varphi_{\pm,g} = \pm m/\lambda$
- $V''(\varphi_{\pm,g}) = (2m)^2$ (Curvature)
- Classical length scale $\xi = \frac{1}{\sqrt{V''(\varphi)}} = \frac{1}{2m}$

- $\varphi_\pm' = \pm \lambda^{1/2}\left(m^2/\lambda - \varphi_\pm^2\right)$

$$\Rightarrow \boxed{\varphi_{\pm,a}(x) = \pm \frac{m}{\lambda} \tanh m(x-a)}$$

$$\Rightarrow \boxed{\varphi_{\pm,a}(x,t) = \pm \frac{m}{\lambda} \tanh m(x-a)}$$

- $\mathcal{H}_{\pm,a}(x,t) = \boxed{\dfrac{m^4}{\lambda \cosh m(x-a)}}$

<u>total</u>

- $H_{\pm,a}(x,t) = \boxed{\dfrac{4m^3}{3\lambda}}$

<u>Note</u> : • a is a free parameter that fixes the center about which energy density is concentrated.

- <u>Tunneling</u>

eg
- from $\varphi_{+,a}(-\infty,t)$ to $\varphi_{-,a}(+\infty,t)$

$-m/\sqrt{\lambda}$ $\qquad$ $+m/\sqrt{\lambda}$

- <u>Energy barrier</u>

$$H = \frac{4m^3}{\lambda}$$

This suppresses the tunneling rate.

QM Probability $\simeq \boxed{e^{-2H}}$

- <u>Boosted</u>

$$\boxed{\varphi_{\pm,a,\beta}(x,t) = \pm\tan\left(\frac{x-a-\beta t}{\sqrt{1-\beta^2}}\right)}$$

- $H(\varphi_{\pm,a,\beta}) = \cosh\varphi\,\frac{4m^3}{\lambda} = \gamma\left(\frac{4m^3}{\lambda}\right)$

- $P_{\pm,a,\beta} = \sinh\varphi\,\frac{4m^3}{\lambda} = (v\gamma)\left(\frac{4m^3}{\lambda}\right)$

- $M = \frac{4m^3}{3\lambda}$ (rest mass)

- Half width $= \frac{\sqrt{1-\beta^2}}{4m}$

- <u>Note</u>

  Boosted soliton still have no dispersion

  The shape is altered by boost.

Sine Gordon Equation

- $\boxed{\varphi_{0,n}(x,t) = 2\pi n}$ (Infinite Degenerate Vaccua)

- $\boxed{\varphi_{\pm,a}(x) = 4 \arctan e^{\pm(x-a)}}$