Graduate Qualifying Examination

Part 1/2: January 11–12, 2016

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet.
- On the cover of each answer booklet put only your *assigned number* and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 180 minutes are alloted for each part.
- **Calculator policy**:
  Use of a graphing or scientific calculator is permitted provided that it has *none* of the following capabilities:

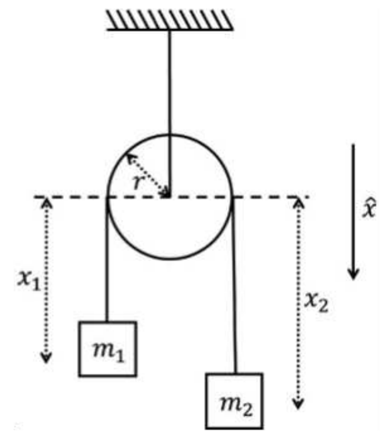
  – programmable
  – algebraic operations
  – storage of ASCII data

  Handheld computers, PDAs, and cell phones are explicitly prohibited.

Part I: Classical Mechanics (Mo, 11-Jan-2016, 2-5 pm)

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

Problem 1. A string of length L connects to masses m_1 and m_2 in an Atwood machine as shown in the figure below, where the pulley and the string are assumed to be massless. The Earth's gravitational field is acting in the x direction. Using the nomenclature used in the figure



(a) Derive the equations of motion for m_1 and m_2, including any constraint conditions.

(b) Calculate the acceleration of the two masses as a function of m_1, m_2 and gravitational acceleration g.

(c) Calculate the tension T in the string.

Problem 2. A force field is given in Cartesian coordinates as

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = (ay^2z^3 - 6bxz^2)\,\hat{\mathbf{x}} + 2axyz^3\,\hat{\mathbf{y}} + (3axy^2z^2 - 6bx^2z)\,\hat{\mathbf{z}}$$

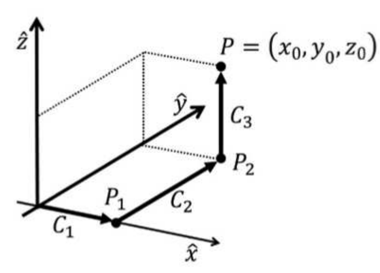
where a and b are constants and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the Cartesian unit vectors.

(a) Determine if the given force field is conservative.

(b) A point-like particle is moved within this force field along a path starting at the origin $O = (0, 0, 0)$ and ending at point $P = (x_0, y_0, z_0)$,

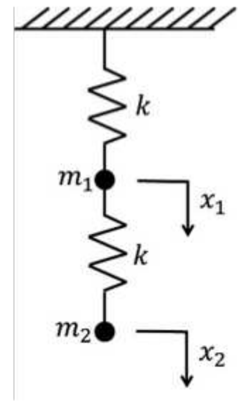
$$O \xrightarrow{C_1} P_1 \xrightarrow{C_2} P_2 \xrightarrow{C_3} P_3,$$

which leads over points P_1 and P_2. The paths C_1, C_2, and C_3 are parallel to the x-, y-, and z-axis, respectively. Parametrize the paths C_1, C_2, and C_3 and calculate the work needed to move the particle from O to P.



(c) Can the force field $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ be derived from a potential? If yes, calculate the corresponding potential.

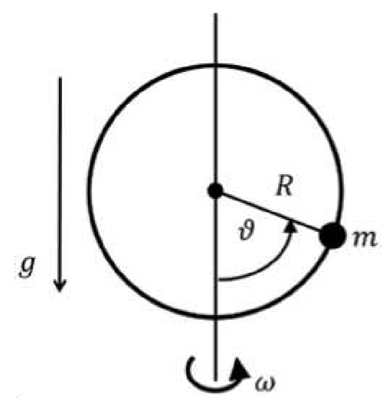
Problem 3. Two masses m_1 and m_2 are coupled by a spring and suspended vertically using a second spring, as shown in the figure below. Both springs are identical and have the spring constant k. In addition you can assume that $m_1 = m_2 = m$.



(a) Write the equations of motion for x_1 and x_2, where x_1 and x_2 denote the displacements from the positions of equilibrium for m_1 and m_2, respectively.

(b) Calculate the eigenfrequencies and eigenmodes of the oscillation.

Problem 4. A point-like bead of mass m glides on a massless and frictionless ring. The ring has a radius R and rotates with constant angular speed ω around its axis in the Earth's gravitational field, as shown in the figure below.



(a) Write down the Lagrangian of the system in terms of the minimum number of independent degrees of freedom.

(b) Determine the Euler-Lagrange equation(s) of motion for the Lagrangian in part (a).

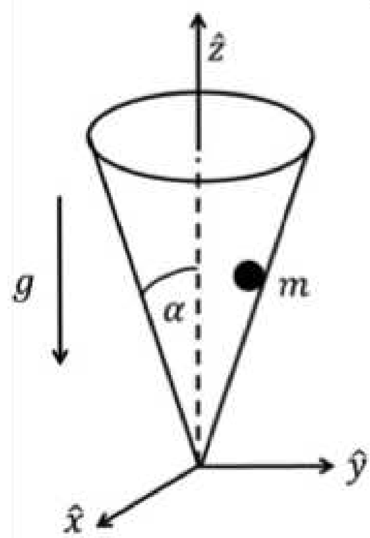
(c) Calculate $\theta(t)$ for $\theta \ll 1$ for $\omega^2 < g/R$.

Problem 5. A point-like bead of mass m is rolling without friction on the inside of a circular cone in the gravitational field of the Earth, as shown in the figure below. The angle between the generatrix and the cone axis is α, i.e., the cone is parameterized by $r = z \tan\alpha$. Using cylindrical polar coordinates, (r, ϕ, z),

(a) Write down the Lagrangian of the system in terms of r and ϕ.

(b) Determine the Euler-Lagrange equations of motion for this Lagrangian.

(c) Are there any cyclic coordinates for the Lagrangian in part (a), and if so, which one(s), and what are the associated quantities?

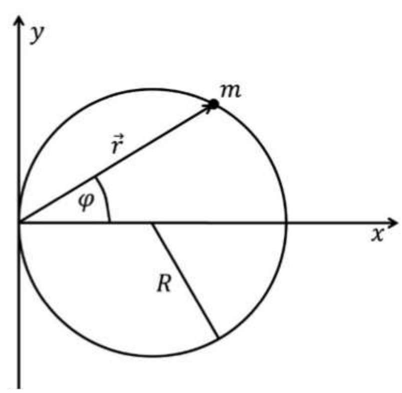


Problem 6. A point-like particle of mass m travels under the influence of a central force $\vec{\mathbf{F}}$, centered at $(0, 0)$, on a circular path with radius R, centered at $(R, 0)$, through the coordinate origin, as shown in the figure below.

(a) Determine the position of the mass by calculating $r = r(\varphi)$, i.e., the radial distance from the origin as a function of the polar angle φ.

(b) Express the total energy in terms of r, $dr/d\varphi$, L, and V (where L and V are the angular momentum and potential, respectively), without any dependencies on $\dot{r}$ or $\dot{\varphi}$.

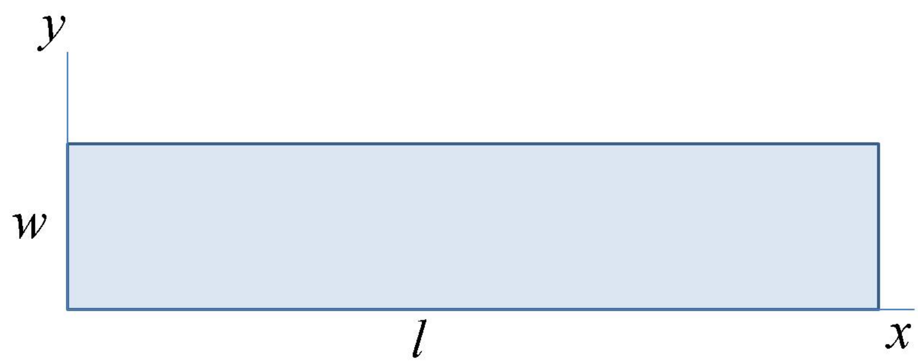
(c) Calculate the magnitude and direction of the force.



Part II: Electricity and Magnetism (Tu, 12-Jan-2016, 2-5 pm)

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

Problem 1. Consider a rectangle of width w (along y) and length l (along x). The corners of the rectangle are at $(x, y) = (0, 0)$, $(0, w)$, $(l, 0)$ and (l, w), as shown in the figure below.



Assume that the two dimensional Laplace equation is valid within this rectangle. Solve the Laplace equation for the following boundary conditions:

(a) $V = V_0 = \text{const.}$ for $x = 0$ on the left, $V = 0$ for $x = l$ on the right, and $\partial V/\partial n = 0$ along the top and bottom ($y = 0$ and $y = w$).

(b) $V = V_0 \cos(\pi y/w)$ for $x = 0$ on the left, $V = 0$ for $x = l$ on the right, and $\partial V/\partial n = 0$ along the top and bottom ($y = 0$ and $y = w$).

Problem 2. A long coaxial cable consists of an inner cylinder with radius a and a thin outer cylindrical shell with radius $b > a$. The region between the cylinders is filled with a dielectric with dielectric constant κ. The charge density function is given by

$$\rho(r) = \frac{C}{4\pi}\left[\frac{10}{a^4}(r-a)\,\Theta(a-r) + \frac{1}{b^2}\,\delta(r-b)\right],$$

where $\Theta(x)$ is the Heaviside step function, $\delta(x)$ is the Dirac delta function, and C is a dimensional constant. Find the electric field in each of the three regions:

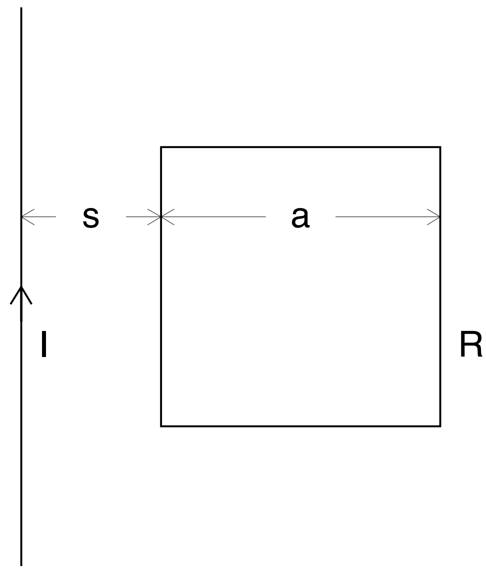
(a) inside the inner cylinder ($r < a$),

(b) between the cylinders ($a < r < b$), and

(c) outside the cable ($r > b$).

Problem 3. An origin-centered spherical shell with radius R and uniform surface charge density σ spins with angular frequency ω about the z-axis. Find the magnetic field at $z = R/2$ along the axis of rotation.

Problem 4. A square loop of side a and resistance R lies at a distance s from an infinite straight wire which carries a current I, in the same plane as that of the loop – as shown in the figure below. The current is turned down to zero in some time Δt.



(a) Determine the orientation of the induced current in the square loop.

(b) Calculate the total charge which passes through any given point of the loop during the time Δt.

Problem 5. Suppose a current density $\vec{J}(\vec{r})$ is constant with time but the charge density $\rho(\vec{r}, t)$ is not. This is a situation which happens when charging a capacitor, for example.

(a) Using the continuity equation, show that the charge density at any particular point is a linear function of time, i.e., $\rho(\vec{r}, t) = A(\vec{r}) + B(\vec{r}) \cdot t$.

(b) Starting from the Biot-Savart and Coulomb laws,

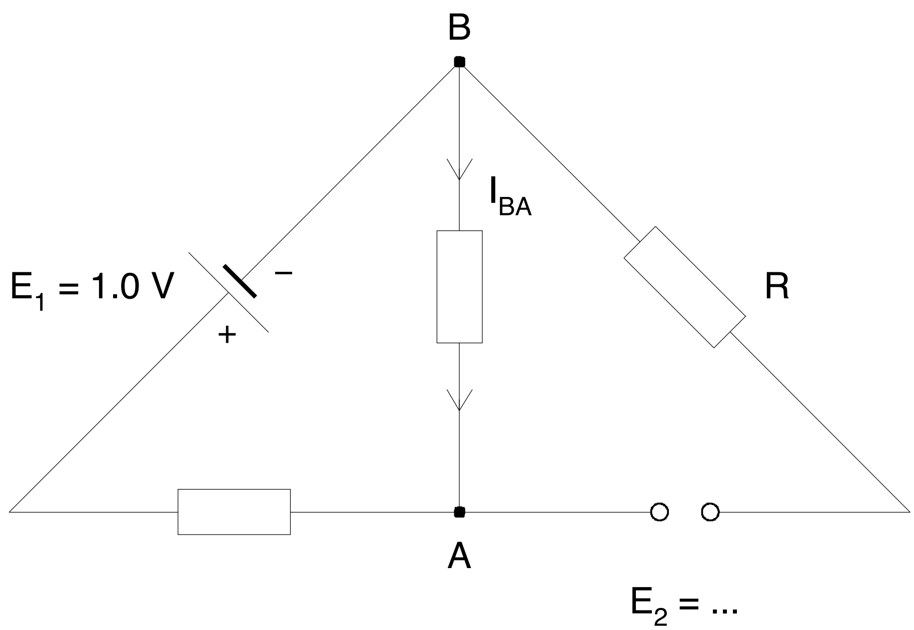
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}\, d^3r' \quad \text{and} \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})\,(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}\, d^3r',$$

respectively, show that $\vec{B}(\vec{r})$ obeys Ampere's law with the displacement current term. You may assume that both current and charge densities vanish for $r \to \infty$.

Note: $\vec{\nabla} \times (\vec{a} \times \vec{b}) = \vec{a}\,(\vec{\nabla} \cdot \vec{b}) - \vec{b}\,(\vec{\nabla} \cdot \vec{a}) + (\vec{b} \cdot \vec{\nabla})\,\vec{a} - (\vec{a} \cdot \vec{\nabla})\,\vec{b}$

Problem 6. Consider the circuit illustrated below, in which all resistors are identical ($R = 1.0\,\mathrm{k\Omega}$) and the two batteries have no internal resistance. Assuming that the current through the middle branch of the circuit is $I_{BA} = 1.0\,\mathrm{mA}$ and flows as indicated, from point B to point A, calculate the EMF of the second battery, $\mathcal{E}_2$, and indicate its orientation (polarity), given that the EMF of the first battery is $\mathcal{E}_1 = 1.0\,\mathrm{V}$.

Note: the batteries EMF in the figure below are labeled by E, as opposed to the standard symbol, $\mathcal{E}$.



Graduate Qualifying Examination

Part 2/2: August 15–16, 2016

General Instructions

- No reference materials are allowed.

- Do *all* your work in the corresponding answer booklet.

- On the cover of each answer booklet put only your *assigned number* and the part number/subject.

- Turn in the questions for each part with the answer booklet.

- 180 minutes are alloted for each part.
  Thermal Physics: 90 minutes (up from 60).

- **Calculator policy**:
  Use of a graphing or scientific calculator is permitted provided that it has *none* of the following capabilities:

  – programmable
  – algebraic operations
  – storage of ASCII data

  Handheld computers, PDAs, and cell phones are explicitly prohibited.

Part I: Quantum Mechanics (Mo, 15-Aug-2016, 2-5 pm)

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

Problem 1. The one-dimensional time-independent Schrödinger equation is given by

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\phi(x) = E\,\phi(x)\,. \tag{1}$$

Consider a particle of mass m in the presence of a one-dimensional finite square well given by the potential $V(x)$.

(a) Let the potential be given by

$$V(x) = \begin{cases} 0 & \text{for} \quad |x| \geq a/2 \\ -V_0 & \text{for} \quad |x| < a/2 \end{cases}, \quad \text{with} \quad V_0 > 0. \tag{2}$$

This potential separates the spatial domain into three regions which we refer to as regions I, II, and III for $x < -a/2$, $-a/2 < x < a/2$, and $x > a/2$, respectively. The general form of the even parity solutions in regions I, II, and III is given by $\phi_I \sim e^{\alpha x}$, $\phi_{II} \sim \cos(\beta x)$, and $\phi_{III} \sim e^{-\alpha x}$. Express α and β in terms of the given quantities (E, V_0, m, $\hbar$). Pick the correct signs.

(b) Derive the relation between α and β by considering the point $x = -a/2$ where the wave functions merge.

(c) Now replace the potential in Eq.(2) above by a potential well in form of a delta potential,

$$V(x) = -\frac{\hbar^2}{2m}\,V_0\,\delta(x),$$

with $V_0 > 0$. The solution $\phi(x)$ has a discontinuity at $x = 0$. Compute the size of the discontinuity, $\phi\big|_{0^+} - \phi\big|_{0^-}$, using equation (1).

(d) Find the energy of the bound state solution of the potential given in part (c) above. Express your result in terms of V_0, $\hbar$, and m.

Problem 2. A particle of mass m and charge q sits in the three-dimensional harmonic oscillator potential

$$V(\mathbf{x}) = \frac{k}{2}\left(x^2 + y^2 + z^2\right),$$

with the real-valued constant k, and $\mathbf{x} = (x, y, z)$. Then the Hamiltonian reads

$$H_0 = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{x}) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(\mathbf{x})$$

.

(a) Using the Heisenberg algebra for position and momentum operators, obtain the commutators for all components of the raising and lowering operators, $[a_\alpha, a_\beta]$ and $[a_\alpha, a_\beta^\dagger]$, where $\alpha, \beta = x, y, z$ and

$$a_x = \left(\frac{m\omega}{2\hbar}\right)^{1/2} (x + \frac{i}{m\omega} p_x) \quad \text{and} \quad a_x^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} (x - \frac{i}{m\omega} p_x) \quad \text{with} \quad \omega = \sqrt{k/m}.$$

(b) What are the energy eigenvalues of the system?

(c) What are the degeneracies of the first three excited states?

(d) At time $t = -\infty$ the oscillator is in its ground state. Now consider adding the time-dependent perturbation $\delta V(t) = qAe^{-(t/\tau)^2} z$, with constants q, A and τ. According to time-dependent perturbation theory, what is the probability for finding the system in an excited state at the time $t = +\infty$? In other words, what is the probability for $|\psi(t = +\infty)\rangle$ not to be the ground state? Provide a result which is accurate to first order in the perturbing potential δV.

Problem 3. Let $|j, m\rangle$ be a simultaneous eigenstate of the squared angular momentum operator $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ with eigenvalue $\hbar^2 j(j+1)$, and of J_z with eigenvalue $\hbar m$.

(a) Given the ladder operators $J_\pm = J_x \pm iJ_y$, what is the expectation value of J_x^2 and J_y^2 in the state $|j, m\rangle$?
Note: There are at least two ways of solving this task. One would involve the relation $J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, |j, m \pm 1\rangle$.

(b) Consider a system with the spin operator $S_x^2 + S_y^2 + S_z^2$, where S_x, S_y and S_z satisfy the angular momentum algebra in part (a) above. What is the expectation value of S_x^2 in a state with spin 1/2, and the spin projected on the z-axis being $-\hbar/2$?

Problem 4.

(a) First, consider a system which is described by a time-independent Hamiltonian H_0. Let $\{|m\rangle\}$ be the orthonormal energy eigenstates with corresponding eigenvalues E_m. Write the probability for finding the system in an arbitrary energy eigenstate $|n\rangle$ at time $t > 0$, for the system initially in one of the following two cases:

  (i) the energy eigenstate $|\psi, t = 0\rangle = |n\rangle$,

  (ii) the state $|\psi, t = 0\rangle = |\psi_0\rangle$, which is not an energy eigenstate.

(b) Now consider the specific example of a two state system, i.e., a system with eigenstates $\{|m\rangle\} = \{|+\rangle, |-\rangle\}$ and eigenvalues $E_+ = 1/2$ and $E_- = -1/2$. Compute the probabilities for finding the system in the state $|\pm\rangle$ at time $t > 0$ given that the initial state is

(i) the eigenstate $|\psi, t=0\rangle = |+\rangle$,

(ii) the state $|\psi, t=0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$.

(c) Let us turn to a time-dependent Hamiltonian given by

$$H(t) = H_0 + V(t), \quad \text{with} \quad H_0 = E_+|+\rangle\langle+| \, + \, E_-|-\rangle\langle-|$$

and

$$V(t) = \gamma\left(e^{i\omega t}|+\rangle\langle-| + e^{-i\omega t}|-\rangle\langle+|\right),$$

where γ is a real-valued constant. Compute the probability for finding this system in the state $|-\rangle$ at time $t=0$, and then in the state $|+\rangle$ at a later time t.

Problem 5. The observable "electron spin" is represented by

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Work in the orthonormal basis of eigenstates of σ_z.

(a) Compute the eigenvectros $|\pm\rangle$ and the corresponding eigenvalues $\lambda_\pm$ of σ_z.

(b) Show that σ_x and σ_y have the same eigenvalues as σ_z. Is the same true for the eigenstates? What does this imply for the observables associated with these matrices, i.e., can one measure S_x and S_y simultaneously?

(c) Write the uncertainty relation for the measurements of the following pairs of matrices: (σ_x, σ_y), (σ_x, σ_z), and (σ_y, σ_z).

Problem 6. The ground state wavefunction of the hydrogen atom is given by

$$\Psi(r) = A\,\exp\left(-r/a_0\right),$$

where a_0 is the Bohr radius and A is the normalization constant.

(a) Find the value of r associated with the maximum probability density.

(b) Find the expectation value of the radial coordinate r.

(c) Find the uncertainty in a measurement of the radial coordinate r.

Note: potentially useful relationship (integration by parts):

$$\int_0^\infty r^n\, e^{-r}\, dr = n\int_0^\infty r^{n-1}\, e^{-r}\, dr.$$

Part II: Thermal Physics (Tu, 16-Aug-2016, 2:00-3:30 pm)

Do any 2 of the 3 problems.
If you try all 3 problems, indicate clearly which 2 you want marked.
If there is no clear indication, the first 2 problems will be marked.

Problem 1. The thermodynamic fundamental relation for a rubber band is given by

$$S(U, L) = S_0 + cL_0 \ln \frac{U}{U_0} - \frac{b}{2(L_1 - L_0)}(L - L_0)^2,$$

where S is the entropy, U is the internal energy, L is the length of the rubber band, L_0 is the unstretched length of the rubber band, and L_1 is its maximum stretched length. In this case L forms an analog of volume.

(a) Express U in terms of the temperature T.

(b) Show that the tension τ in the rubber band is given by

$$\tau = bT\frac{L - L_0}{L_1 - L_0}$$

by equating the work done to the change in internal energy obtained by adiabatically stretching the rubber band.

(c) The rubber band is stretched by an amount dL at constant T. Calculate the heat transfer dQ into the rubber band. How is this related to the work done?

Problem 2. A tank has a volume of $0.1\,\mathrm{m}^3$ and is filled with He gas at a pressure of 5×10^6 Pa. A second tank has a volume of $0.15\,\mathrm{m}^3$ and is filled with He gas at a pressure of 6×10^6 Pa.

(a) A valve connecting the two tanks is opened. Assuming He to be a monoatomic ideal gas and the walls of the tanks to be adiabatic and rigid, find the final pressure of the system.

(b) If the temperatures within the two tanks before opening the valve had been $T =$ 300 K and 350 K respectively, what would the final temperature be?

(c) If the first tank had contained He at an initial temperature of 300 K, and the second had contained a diatomic ideal gas with $c_P/c_V = \gamma = 5/2$ at an initial temperature of 350 K, what would the final temperature be?

Problem 3.

(a) A system is composed of two quantum wells, each having an energy spectrum given by $n\hbar\omega_0$, where $n = 1, 2, \ldots$ The total energy of the system is $E_1 = k_1\hbar\omega_0$, where k_1 is a positive integer. How many microstates are available to the system with energy E_1? What is the entropy in terms of E_1?

(b) What is the temperature of the system in terms of E_1 in the large-energy limit?

(c) Consider a second system, similar to the first, except that the spacing between two consecutive energy levels in each of the wells is twice as high, i.e., $\omega_0 \to 2\omega_0$. Assuming that the two systems are in thermal equilibrium with each other, calculate the entropy of the combined system in terms of the total energy E_{tot} and ω_0 in the large-energy limit.
