Graduate Qualifying Exam

Department of Physics & Astronomy, University of Alabama

6 January 2015 and 18 August 2015

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet.
- On the cover of each answer booklet, make sure to write your *assigned number* and the part number/subject.
  Exams are graded anonymously, so do not write your name.
- Turn in the question sheet for each part with the answer booklet.
- 120 minutes are allotted for each part, except for Thermal Physics (60 minutes).
- **Calculator policy**: Use of a graphing or scientific calculator is permitted provided that it has *none* of the following capabilities:
  - programmable
  - algebraic operations
  - storage of ASCII data

  Handheld computers, PDAs, and cellphones are explicitly prohibited.

Part I: Electricity and Magnetism

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. In a region of space free of currents, the electric and magnetic fields at $t = 0$ are determined to be:

$$\vec{E}(\vec{r}, 0) = \alpha x^2 \ \hat{j}$$
$$\vec{B}(\vec{r}, 0) = 0$$

where α is a constant and $\hat{j}$ is a unit vector in the y-direction.

   (a) What is the charge density at $t = 0$, $\rho(\vec{r}, 0)$?

   (b) Calculate the magnetic field as a function of time.

   (c) Calculate the electric field as a function of time.

2. A spherical charge distribution has a charge density given by $\rho(r) = \rho_0 e^{-r/\lambda}$ and a total charge of Q.

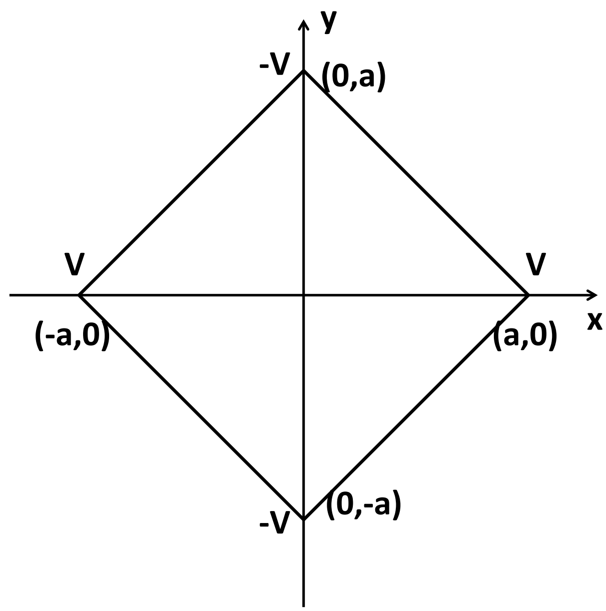
   (a) Find ρ_0 in terms of Q and λ.

   (b) Use Gauss's law to find the magnitude of the electric field, $E(r)$, and show that is has the correct form for $r \gg \lambda$, where λ is the characteristic length of the charge distribution.

   (c) Suppose we add a charge of magnitude $2Q$ and opposite sign at the origin. How does this change the electric field?

Hint: Integrals can be done by integration by parts.

3. A hollow square has its corners at $(a, 0)$, $(0, a)$, $(-a, 0)$, and $(0, -a)$, as shown in the figure. The electric potential along the sides of the box is symmetric about both the x and the y-axes. The corners on the x-axis are at a potential V and the corners on the y-axis are at a potential $-V$.



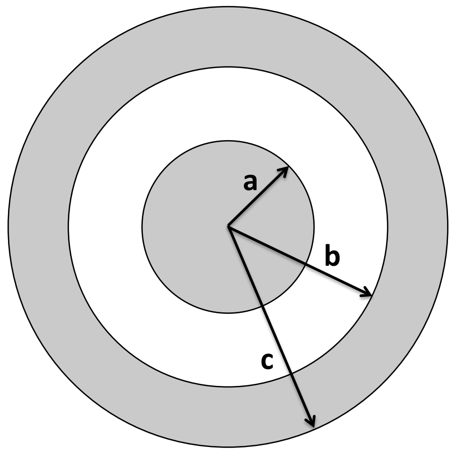
Start with the following trial solution for the potential:

$$\phi(x, y) = A + Bx + Cy + Dx^2 + Ey^2 + Fxy$$

(a) Use the symmetry properties of the boundary conditions, and the Laplacian, to find the values of the constants A, B, C, D, E, and F, and write down the explicit form of the potential inside the box in terms of V, a, x, and y.

(b) From the potential, calculate the electric field inside the box.

4. The figure below shows the cross-section of a coaxial structure consisting of an inner conductor with radius a and an outer conducting shield of inner radius b and outer radius c. A total current I flows upward (out of the page) through the inner conductor and returns downward (into the page) through the outer shield. Assume the current density is uniform in both conductors.

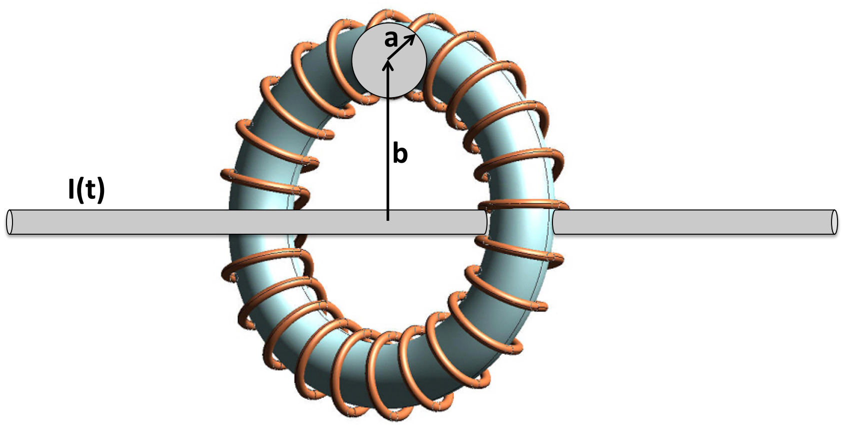


(a) Find the magnetic field (magnitude and direction) as a function of radial distance, $\vec{B}(\rho)$.

(b) Sketch the magnitude of the magnetic field as a function of radial distance.

(c) How would the solution be modified if a current $2I$ flows upward through the inner conductor while the returning downward current remains I?

5. A Rogowski Coil is constructed of a soft iron (magnetic permeability μ) torus of mean radius b and a circular cross-section of radius $a = b/8$. It is wound along its entire diameter with a thin wire such that the number of turns per unit length is n. A wire passing through the hole in the torus carries a time-varying current $I(t) = I_0 \cos(\omega t)$. (You may assume that the current changes slowly enough that the quasistatic approximation holds.)



   (a) Find the magnetic field strength, $\vec{H}(t)$, inside the torus.

   (b) Find the induced EMF in the coil.

6. Consider a source-free, *conductive* medium with constant permittivity ϵ and permeability μ.

   (a) Assuming that the medium is ohmic, with a constant conductivity σ, write down the differential form of Maxwell's equations for the electric field $\vec{E}$ and the magnetic field $\vec{B}$, in such a medium.

   (b) Derive the wave equation for the electric field in such a medium.

   (c) Solutions to this equation are damped plane waves. Consider such a wave traveling in the positive z-direction, with an electric field given by

$$\vec{E}(z,t) = E_0 \mathrm{e}^{i(\alpha z - \omega t)} \mathrm{e}^{-\beta z} \ \hat{i}$$

   where α and β are real. Find the damping factor β in terms of ω, σ, ϵ, and μ.

Recall that for a vector field $\vec{A}$:

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Part II: Quantum Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want graded.
If there is no clear indication, the first 5 attempted problems will be graded.

1. We consider a 1-dimensional quantum mechanical system with the Hamiltonian,

$$H(t,x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(t,x),$$

where $V(t,x)$ is a real function of the time t and and the position x. The wavefunction Ψ obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t,x) = H(t,x)\Psi(t,x).$$

(a) For the wavefunction, we define the probability density (ρ) and the current (J) as

$$\rho(t,x) = \Psi^*\Psi \quad \text{and} \quad J(t,x) = \frac{\hbar}{2im}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right).$$

Show that

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0,$$

and explain the physical significance of this relationship.

(b) When the potential of the system V is time-independent, we decompose the wave function as $\Psi(t,x) = f(t)u(x)$. Derive the time-independent Schrödinger equation (eigenvalue equation) for $u(x)$.

(c) When the Hamiltonian is time-independent, derive the equation

$$\left(\hbar^2\frac{\partial^2}{\partial t^2} + H^2\right)\Psi(t,x) = 0.$$

(d) By applying the Einstein relation in the theory of relativity to the Hamiltonian squared, $H^2 = p^2c^2 + m^2c^4$, where p is the momentum operator, and m is the mass of the particle, derive the relativistic wave equation satisfied by $\Psi(t,x)$.

2. In a 2-dimensional system, an electron of mass m is trapped in an infinite potential well,

$$V(x,y) = \begin{cases} 0 & \text{for } 0 \le x \le L \text{ and } 0 \le y \le L \\ \infty & \text{otherwise} \end{cases}$$

(a) Write the solution of the time-independent Schrödinger equation in the region inside the well.

(b) Determine the energy spectrum of the system.

(c) Find the wavelength of the absorbed photon when the electron is excited from the ground state to the 2nd excited state.

3. A particle of mass m propagates in the positive x-direction under the influence of the potential

$$V(x) = \begin{matrix} 0 & (x \leq 0) \\ -V_0 & (x > 0) \end{matrix} ,$$

where $V_0 > 0$, and the energy of the particle is $E > 0$.

(a) Express the wavefunction in the region $x \leq 0$ in terms of the given quantities.

(b) Express the wavefunction in the region $x > 0$ in terms of the given quantities.

(c) Calculate the transmission coefficient, T, and express it in terms of E and V_0.

(d) Find the energy of the particle which gives $R = \frac{1}{4}$, where R is the reflection coefficient.

4. In quantum mechanics, the spin operators are defined as

$$S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y, \quad S_z = \frac{\hbar}{2}\sigma_z,$$

where $\sigma_{x,y,z}$ are Pauli matrices given by

$$\sigma_x = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} , \qquad \sigma_y = \begin{matrix} 0 & -i \\ i & 0 \end{matrix} , \qquad \sigma_z = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} .$$

(a) Verify that the spin operators satisfy the following algebra,

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y,$$

(b) Defining the total spin operator as $S^2 = S_x^2 + S_y^2 + S_z^2$, verify $[S^2, S_z] = 0$.

(c) The relation $[S^2, S_z] = 0$ (more generally, finding independent operators which commute with each other) is of special importance in quantum mechanics. Explain why.

(d) Consider a composite state which consists of two spin-1/2 particles. What is the spin of this composite state? Consider all possibilities.

5. We consider a system with only two states based on the Hamiltonian,

$$H_0 = E_0 \begin{matrix} -1 & 0 \\ 0 & 2 \end{matrix} ,$$

with $E_0 > 0$.

(a) Determine the two normalized eigenstates in the form of column vectors and the corresponding eigenvalues.

(b) If the system is perturbed so that the Hamiltonian is $H = H_0 + H_1$, where

$$H_1 = \epsilon \begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} \tag{1}$$

with $0 < \epsilon \ll E_0$, what are the energy eigenvalues in the first order perturbation with respect to $\epsilon/E_0 \ll 1$.

(c) Diagonalizing the total Hamiltonian, find the exact eigenvalues.

(d) Expand the exact solutions of part (c) with respect to $\epsilon/E_0 \ll 1$ and verify the results obtained in part (b).

6. Consider a quantum mechanical system with three possible orthogonal states, A, B and C given by

$$|\psi_A\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad |\psi_B\rangle = \begin{pmatrix} -\sqrt{\frac{3}{8}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad |\psi_C\rangle = \begin{pmatrix} \sqrt{\frac{3}{8}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

In this system, the normalized energy eigenstates are

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and the corresponding eigenvalues are given by $E_1 = E_0$, $E_2 = 2E_0$ and $E_3 = 3E_0$ with $E_0 > 0$.

(a) If the wavefunction at time $t = 0$ is found to be $|\psi(0)\rangle = |\psi_C\rangle$, express the wavefunction $|\psi(t)\rangle$ at an arbitrary time t.

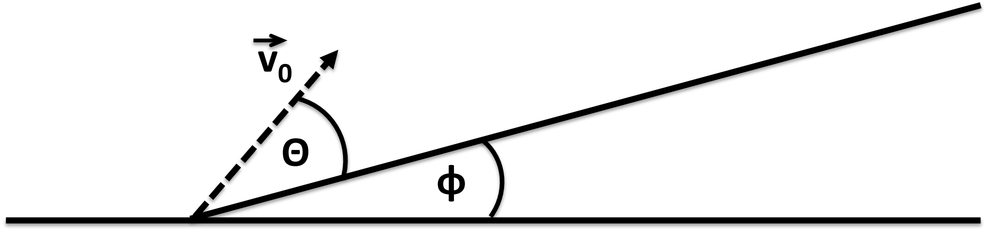
(b) For the state in part (a), express the probability to find the state A at a time $t > 0$.

(c) For the state in part (a), what is the probability to find the state B at a time $t = \frac{\pi\hbar}{2E_0}$.

Part III: Classical Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. A plane is inclined at an angle ϕ to the horizontal. At time $t = 0$, a ball is thrown from the bottom of the incline with an initial speed v_0 at an angle of θ above the plane, with $\theta + \phi < \pi/2$. Choose your axes to have x parallel and y perpendicular to the plane. (You may neglect air resistance in this problem.)



   (a) Find the position of the projectile at time t (prior to it making contact with the plane again).

   (b) When the projectile makes contact with the plane again, show that it lands a distance from its launch point given by:

$$R = \frac{2v_0^2 \sin\theta \cos(\theta + \phi)}{g \cos^2 \phi}$$

   (c) For fixed v_0 and ϕ, find the launch angle which produces the maximum range along the plane, and show that this angle is given by:

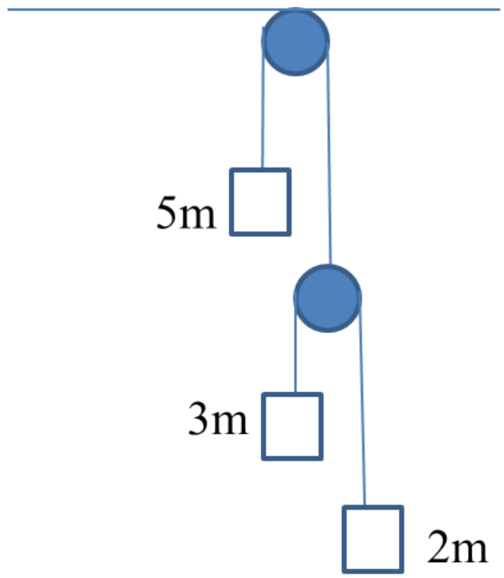
$$\theta_{\max} = \frac{\pi}{4} - \frac{\phi}{2}$$

You may find the following trigonometric identities useful in this problem:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

2. A double Atwood's machine is constructed as shown in the diagram below. The top pulley is fixed and a string of length L_1 is passed over it. One end of this string is attached to a mass $5m$, while the other end supports a second pulley. A string of length L_2 is passed over this second pulley, with a mass of $3m$ at one end and $2m$ at the other end. Assume both pulleys are frictionless and massless.



Choose a suitable pair of generalised coordinates and construct the Lagrangian for this system. Find the equations of motion for the three masses and solve them to find the acceleration (magnitude and direction) of the $5m$ mass.

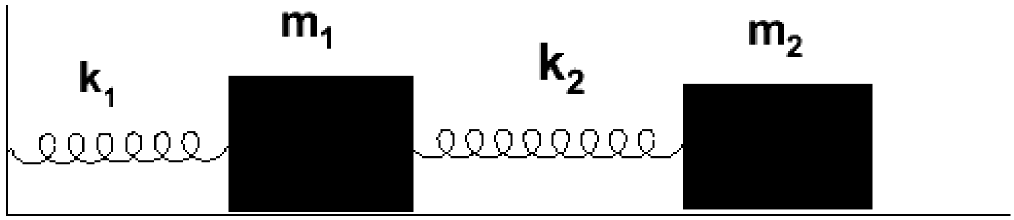
3. Two equal masses m are connected by a massless string of length L that passes through a hole in a frictionless horizontal table, such that one mass slides around on the table, while the other mass hangs below the table, and can move vertically up or down.

   (a) Using as generalised coordinates r and ϕ, the polar coordinates describing the position of the mass on the table, construct the Lagrangian for this system and find the equations of motion.

   (b) Show that the ϕ equation implies the conservation of angular momentum. Express $\dot{\phi}$ in terms of the angular momentum ℓ and use this to eliminate $\dot{\phi}$ from the radial equation, such that the radial equation becomes a function only of r (and its derivatives).

   (c) Now show that there exists a solution where the mass on the table moves in a circular path with constant radius r_0, and find r_0 in terms of l, m, and g.

4. As shown in the diagram below, two masses are joined by two springs, with one spring connected to a wall. The masses are free to slide on a perfectly frictionless horizontal surface. Let x_1 and x_2 be the displacement of each mass from its equilibrium position.



Consider the case where both springs have the same spring-constant, $k_1 = k_2$, and the masses are equal, $m_1 = m_2$. Find the frequency of oscillation and the relative amplitudes and phases of the motion for the normal modes of this system.

5. A boat (mass m) is launched on a lake with initial speed v_0. It moves in a straight line, but experiences a slowing force due to the water of $F(v) = -\alpha e^{\beta v}$, where α and β are constants with appropriate dimensions.
   (a) Find the speed of the boat as a function of time, $v(t)$.
   (b) Find the time taken before the boat comes to rest.
   (c) Find the distance travelled by the boat before it comes to rest, and show that this distance is:
   $$d = \frac{m}{\alpha\beta^2}\left[1 - (1 + \beta v_0)e^{-\beta v_0}\right]$$

Note: you may find the following standard integral useful:

$$\int \ln x \, dx = x \ln x - x$$

6. A rectangular block of mass m, uniform density ρ_B, cross-sectional area A, and height h, is floating in a liquid of uniform density ρ_L. The block is oriented such that the cross-sectional area A is parallel to the liquid surface.
   (a) At equilibrium, what fraction of the block's volume is below the liquid level?
   (b) Find the period of small oscillations of this floating block about its equilibrium position.

Part IV: Thermodynamics

Do any 2 of the 3 problems.
If you try all 3 problems, indicate clearly which 2 you want graded.
If there is no clear indication, the first 2 attempted problems will be graded.

1. A small hole is put in a container which holds a monatomic ideal gas. Particles of mass m exit the container (to vacuum) with a speed distribution of

$$P(v) = C\, v^3\, \mathrm{e}^{-\frac{mv^2}{2kT}}\,,$$

where T is temperature, k is Boltzmann's constant and C is a normalization constant. You may find the following integral useful:

$$\int_0^\infty x^n\, \mathrm{e}^{-x^2}\, dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$$

where $\Gamma(m) = (m-1)!$ for integer values of m.

(a) What value of C is required for $P(v)$ to be a proper probability density?

(b) What is the most probable value of the speed of the particles leaving the container?

(c) What is the rms speed of the particles leaving the container?

(d) What is the mean kinetic energy of the particles leaving the container?

(e) Given that the gas of N particles in the container has an initial temperature of T_i, show that when an infinitesimal number dN of particles leave the container, the temperature of the gas goes down by an infinitesimal amount

$$dT = \frac{1}{3}\frac{dN}{N}T_i$$

(f) If the initial number of particles in the container is N and the initial temperature of the gas is T_i, what is the final temperature of the gas in the container after half of the particles have leaked out?

2. A toy Stirling engine operates on n moles of air between the temperatures of T_L and T_H. The Stirling cycle consists of the following four stages:

i. An isothermal (constant temperature) compression at temperature T_L from volume V_1 to volume V_2

ii. An isochoric (constant volume) heating at volume V_2 from temperature T_L to temperature T_H

iii. An isothermal (constant temperature) expansion at temperature T_H from volume V_2 to volume V_1

iv. An isochoric (constant volume) cooling at volume V_1 from temperature T_H to temperature T_L

You can assume that air is an ideal, diatomic gas.

(a) Sketch the p-V diagram for this cycle.

(b) Find an analytic expression (in terms of the given quantities and the gas constant R) for the net work done by the gas per cycle?

(c) Find an analytic expression (in terms of the given quantities and the gas constant R) for the heat *absorbed* by the gas per cycle?

(d) If the engine operates between 17°C and 67°C, and $V_1/V_2 = 2$, what is its efficiency? How does it compare to the theoretical maximum efficiency of an engine operating under the same physical conditions?

3. Consider a system of N classical, distinguishable, and non-interacting particles. The states of a single particle have energy $\epsilon_n = n\epsilon$ and are n-fold degenerate, with $\epsilon > 0$ and $n = 1, 2, 3, \ldots$. The system is in contact with a thermal reservoir at temperature T.

(a) Find the partition function for this system. *Hint: The following series, and its derivatives, may be of use:*

$$\sum_{1}^{\infty} x^n = \frac{1}{1-x}$$

(b) Obtain an expression for the internal energy per particle as a function of temperature.

(c) Obtain an expression for the entropy per particle as a function of temperature. What is the value of the entropy per particle in the limit of high temperature ($T >> \epsilon/k_{\mathrm{B}}$)?