Graduate Qualifying Exam

Department of Physics & Astronomy, University of Alabama

10-11 January 2013

General Instructions

- No reference materials are allowed.
- Do *all* your work in the corresponding answer booklet.
- On the cover of each answer booklet, make sure to write your *assigned number* and the part number/subject. Exams are graded anonymously, so do not write your name.
- Turn in the question sheet for each part with the answer booklet.
- 120 minutes are allotted for each part, except for Thermal Physics (60 minutes).
- **Calculator policy**:
  Use of a graphing or scientific calculator is permitted provided that it has *none* of the following capabilities:
  - programmable
  - algebraic operations
  - storage of ASCII data

  Handheld computers, PDAs, and cellphones are explicitly prohibited.

Part I: Electricity and Magnetism

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. In the ground state of the H-atom the nuclear charge can be treated in first approximation as a point charge centered at the origin and an electron density of:

$$\rho_e(\vec{r}) = -\frac{e}{\pi a^3} \exp\left(-\frac{2r}{a}\right)$$

Here a is the Bohr radius, $r = |\vec{r}|$, and e is the elementary charge.

(a) Determine the electric field strength E and the potential Φ as a function of r.

(b) Discuss the two limiting cases $r \ll a$ and $r \gg a$.

Hint: you may find the following useful:

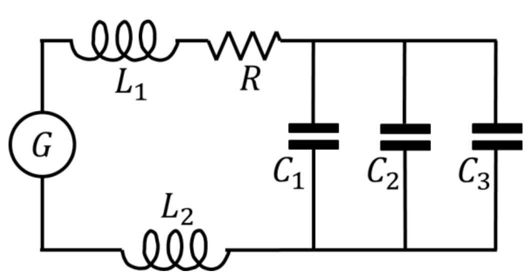
$$\int_0^R x^n e^{-\beta x}\, \mathrm{d}x = \left(-\frac{d}{d\beta}\right)^n \int_0^R e^{-\beta x}\, \mathrm{d}x$$

2. Two concentric metal shells with radii R_1 and R_2 ($R_1 < R_2$) have electric potentials Φ_1 and Φ_2 respectively. Determine the potential $\Phi(r)$ everywhere in space.

Hint: The Laplace operator in spherical coordinates is given by:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

3. In the figure shown below, a generator G with an adjustable frequency is connected to a resistor with resistance $R = 100\ \Omega$, two inductors with inductances $L_1 = 1.7$ mH and $L_2 = 2.3$ mH, and three capacitors with capacitances $C_1 = 4.0\ \mu$F, $C_2 = 2.5\ \mu$F and $C_3 = 3.5\ \mu$F.



(a) Determine the resonance frequency of the given circuit.

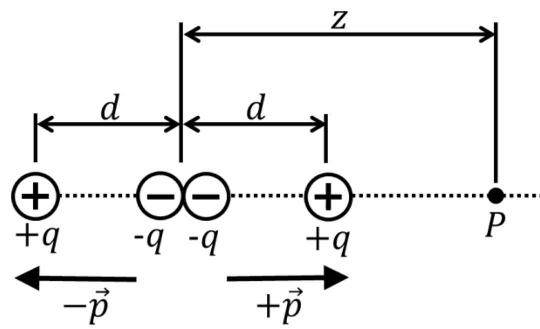
(b) Explain briefly what happens to the resonance frequency if the resistance R is increased.

(c) Explain briefly what happens to the resonance frequency if the capacitor with capacitance C_3 is removed from the circuit.

4. The figure below (which is not to scale) shows an electric quadrupole. It consists of two dipoles with dipole moments that are equal in magnitude but opposite in direction. Show that the magnitude of the electric field E on the axis of the quadrupole for a point P at a distance $z \gg d$ from its center is given by:

$$E = \frac{3Q}{4\pi\epsilon_0 z^4}$$

Here $Q = 2qd^2$ is the quadrupole moment of the charge distribution.



5. The current density $\vec{J}$ inside a long, solid, cylindrical wire of radius $a = 3.1$ mm is in the direction of the central axis, and its magnitude varies linearly with radial distance r from the axis according to:

$$J(r) = J_0 \frac{r}{a}$$

where $J_0 = 310\ \text{A/m}^2$.

Derive an expression for the magnitude of the magnetic field as a function of r for $0 \leq r \leq \infty$ and calculate the magnitude at (a) $r = 0$, and (b) $r = a$.

6. Show by using Maxwell's equations, that in the presence of a charge density $\rho(\vec{r}, t)$ and current density $\vec{j}(\vec{r}, t)$, the vector fields $\vec{E}$ and $\vec{B}$ in vacuum fulfill the inhomogeneous wave equations:

$$ {}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad \vec{E} = \vec{\lambda}_1(\vec{r}, t) $$

$$ {}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad \vec{B} = \vec{\lambda}_2(\vec{r}, t) $$

Determine the functions $\vec{\lambda}_1(\vec{r}, t)$ and $\vec{\lambda}_2(\vec{r}, t)$ in terms of $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$.

Hint: you may find the following relation useful:

$$ \times (\quad \times \vec{A}) = \quad (\quad \cdot \vec{A}) - \quad {}^2\vec{A} $$

Part II: Quantum Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. A positron is a particle with the same mass as an electron but with opposite charge. Electrons and positrons can form bound states called positronium in which the electron and positron orbit about a common center of mass. Assuming the particles travel in circular orbits, use the Bohr quantization condition on the angular momentum to calculate the energy levels for positronium. Given that the energy of the ground state of the hydrogen atom is 13.6 eV, and that the Bohr radius of the hydrogen ground state is 0.053 nm, deduce the values of the ground state energy (in eV) and radius (in nm) for positronium.

   (Note: the mass of the electron is 0.511 MeV/c^2 and the mass of the proton is 938 MeV/c^2)

2. At $t = 0$ a particle of mass m in a harmonic oscillator potential (with frequency ω) is in the initial state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|\psi_0\rangle + \frac{2}{\sqrt{5}}|\psi_1\rangle$$

   where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the normalized eigenfunctions for the ground state and first excited state, respectively:

$$\langle y|\psi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp \frac{-y^2}{2}$$

$$\langle y|\psi_1\rangle = \sqrt{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} y \exp \frac{-y^2}{2}$$

   where $y = \sqrt{\frac{m\omega}{\hbar}}x$

   Hint: in the following question, you may find the following integrals useful:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} \mathrm{d}x = \left(-\frac{d}{d\alpha}\right)^n \int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

   (a) Find the expectation value of the energy in the state $|\psi(0)\rangle$.

   (b) Find $|\psi(t)\rangle$. Is this a stationary state? Explain your answer.

(c) Evaluate the expectation value $\langle \psi(t)|x|\psi(t)\rangle$. What is the frequency of oscillation of this expectation value?

3. Consider the wave function:

$$\psi(x,t) = A\, \mathrm{e}^{-\lambda\,|x|}\, \mathrm{e}^{-i\omega t}$$

(a) Normalize $\psi(x,t)$ to determine the value of the coefficient A.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation, σ, of x. Sketch the graph of $|\psi|^2$ as a function of x and mark approximately the points $(< x > +\sigma)$ and $(< x > -\sigma)$. What is the probability that the particle will be found outside of this range?

4. The y-component of the spin operator is

$$S_y = \frac{\hbar}{2} \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \quad .$$

(a) Find the eigenvectors of this operator and their corresponding eigenvalues.

(b) An electron is in the spin state

$$\chi = \frac{1}{\sqrt{3}} \begin{matrix} \sqrt{2} \\ -1 \end{matrix} \quad .$$

If a measurement of the y-component of the spin is made, what is the probability of finding a value of

$$+\frac{1}{2}\hbar\,?$$

5. A particle of mass m is in an asymmetrical one-dimensional infinite square-well (for $0 \leq x \leq L$) with a perturbation term $V_I(x)$:

$$V_I(x) = \begin{cases} V_0\,, 0 < x < L/2 \\ 0\,, x < 0 \text{ and } x > L/2 \end{cases}$$

where V_0 is small compared to the unperturbed energies of this system.

(a) First determine the energy levels and wavefunctions for the *unperturbed* states (i.e. the asymmetrical one-dimensional infinite square-well) .

(b) Now using perturbation theory, calculate to first order the energy values of the states once the perturbing potential has been applied.

(c) Determine the first-order corrected wavefunction for the ground state of the perturbed system.

Hint: you may find the following trig identities useful:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

6. In scattering theory, the differential cross section as a function of the scattering angle θ can be expressed as:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

where the scattering amplitude $f(\theta)$ can be expanded in terms of partial waves as:

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$$

where δ_ℓ is the phase shift for the ℓth partial wave and $P_\ell(\cos\theta)$ are the Legendre polynomials, of which the first few are:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$\ldots$$

These are obtained from the recursion relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

In the scattering of a particle of energy $E = \hbar^2 k^2/2m$ by a nucleus, an experimenter finds a differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left(0.86 + 2.55\cos\theta + 2.77\cos^2\theta\right)$$

(a) What partial waves are contributing to the scattering, and what are their phase shifts at the given energy?

(b) Find the total cross section.

Part IIIa: Classical Mechanics

Do any 5 of the 6 problems.
If you try all 6 problems, indicate clearly which 5 you want marked.
If there is no clear indication, the first 5 problems will be marked.

1. A block of wood of mass m slides on a horizontal surface that has been lubricated with a particular oil such that the block suffers a viscous resistance that is proportional to the speed to the power 3/2, namely $f(v) = -cv^{3/2}$.

   (a) Calculate the dimensions of the proportionality constant c.

   (b) Assuming that the initial speed of the block is v_0 at $t = 0$, calculate $v(t)$ and describe the motion. In particular, specify what happens to $v(t)$ in the limit of $t \to \infty$.

   (c) Assuming that the block starts from $x_0 = 0$ at $t = 0$, calculate its position as a function of time, $x(t)$. In particular, calculate $x(t)$ in the limit of $t \to \infty$.

2. Determine whether each of the forces below is conservative or non-conservative. For those which are conservative, find the corresponding potential energy U.

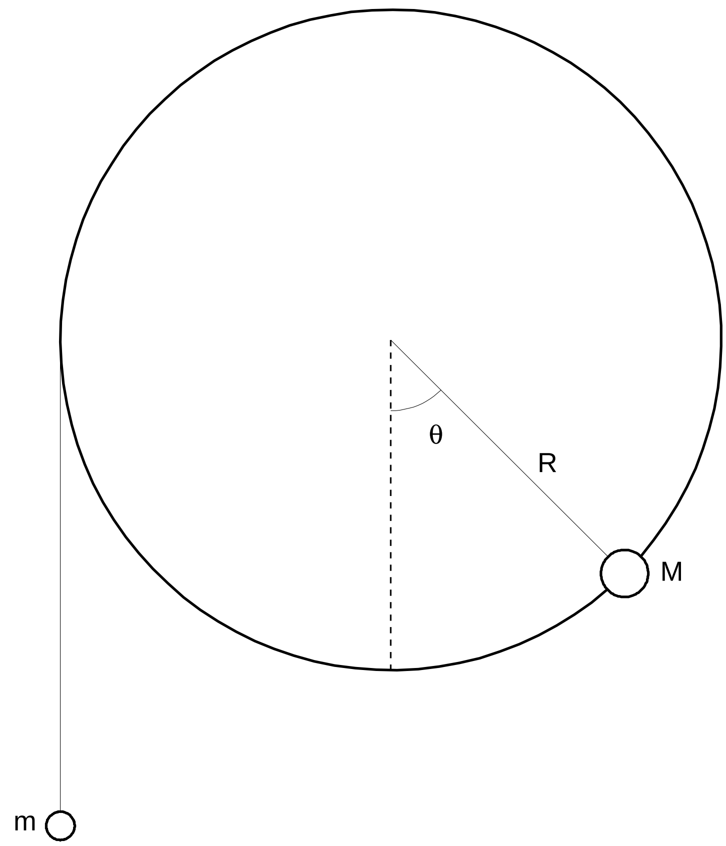
   (a) $\mathbf{F} = k\,(3x, 2y, z)$,

   (b) $\mathbf{F} = k\,(-z, 0, x)$,

   (c) $\mathbf{F} = k\,(x, z, y)$,

   where k is a constant of appropriate dimensions.

3. The figure below shows a massless wheel of radius R, which is free to rotate around a frictionless, horizontal axle. A point mass M is glued to the edge of the wheel, and another point mass m hangs from a massless string wrapped around the perimeter of the wheel.



(a) Obtain the total potential energy of the system of two masses (m and M) as a function of the angle θ (measured with respect to the vertical). Assume that for $\theta = 0$ both masses are at the same height, which also defines the reference point for the potential energy.

(b) Use this to find the values of m and M for which there are any positions of equilibrium and discuss their stability.

4. A particle of mass m experiences a restoring force which is proportional to its displacement from the equilibrium position, $-kx$, and a retarding force which is proportional to its velocity, $-c\dot{x}$, where k and c are the respective proportionality constants.

   (a) Write down the equation of motion for this particle.

   (b) Assuming that the particle starts at its position of maximum displacement, show that in the underdamped case, the solution has the general form: $x(t) = Ae^{-\beta t}\cos(\omega_1 t)$, and find the parameters β and ω_1 in terms of the physical parameters m,k,c. Give the condition for underdamping.

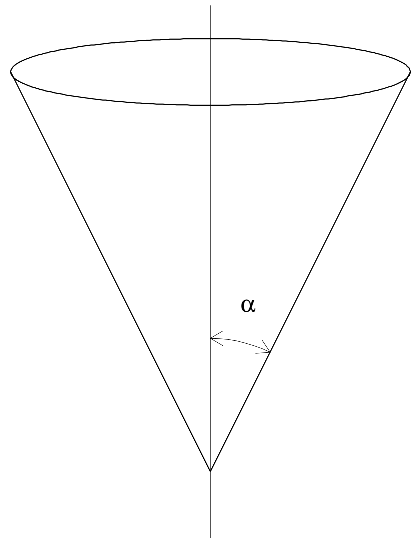
   Now consider a specific case of underdamped motion. Suppose that an undamped oscillator has a natural period τ_0. When a damping force is added, the new period of the underdamped motion is τ_1 and it is found that in the time interval τ_1, the amplitude decays to $1/e$ of its initial value.

   (c) Find an expression for the damping factor β in terms of the *undamped* frequency ω_0.

   (d) Calculate the ratio of the damped to undamped periods, τ_1/τ_0.

   (e) What fraction of the total initial energy has been dissipated through friction during this time interval of τ_1?

5. A particle is confined to move on the inner surface of a circular cone with its axis on the vertical z-axis, vertex at the origin (pointing down), and half-angle α.

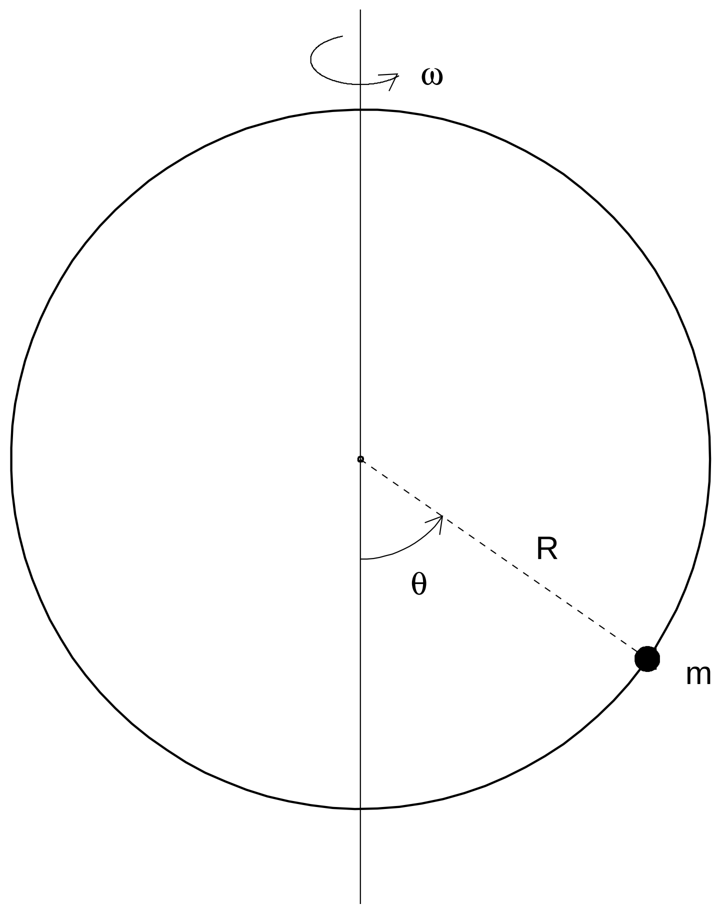


(a) Obtain the Lagrangian for this system in terms of the spherical coordinates r (the radial distance of the particle from the origin) and azimuthal angle ϕ.

(b) Find the two equations of motion. Interpret the ϕ equation in terms of the z-component of angular momentum ℓ_z and show that ℓ_z is conserved. Use this to eliminate $\dot{\phi}$ from the r equation in favor of the constant ℓ_z.

(c) Find the value r_0 of r at which the particle can remain in a horizontal circular path.

6. A bead of mass m is threaded on a frictionless circular wire hoop of radius R. The hoop lies in a vertical plane, and is forced to rotate about its vertical diameter with constant angular velocity $\dot{\phi} = \omega$, as shown in the figure below. The bead's position on the hoop is specified by the angle θ measured up from the vertical. Obtain the Lagrangian for the system in terms of the generalized coordinate θ and find the equation of motion for the bead. Find any stationary solutions, i.e. points where the bead is in equilibrium at some angle.



Part IIIb: Thermal Physics

Do any 2 of the 3 problems.
If you try all 3 problems, indicate clearly which 2 you want marked.
If there is no clear indication, the first 2 problems will be marked.

1. Say that the entropy for some thermodynamic system is given by the function

$$S(N, E, V) = Nk \log(V/V_0) + \sqrt{N\alpha E}\ ,$$

where α and V_0 are constants.

   (a) Is the ideal gas formula $PV = NkT$ valid for this system?

   (b) Obtain the expression for the chemical potential of this system as a function of N, T and V.

   (c) By what factor does the number of accessible states increase when the temperature is doubled (while N and V are held fixed)?

2. Two identical ideal gases are placed in two chambers which have a common wall. Gas A has 10^{23} particles with an initial temperature of 300 ^oK and gas B has 3×10^{23} particles with an initial temperature of 500 ^oK. Both gases are initially at one atmosphere of pressure.

   (a) What is the ratio of the volume of B to the volume of A?

   (b) If heat can pass through the common wall (and nowhere else), what is the final temperature of the gases after they reach thermal equilibrium?

   (c) What are the final pressures of the two gases after they reach thermal equilibrium?

   (d) Say that a crack is created in the common wall as a result of the pressure difference, which then allows particles to flow from one chamber to the other. What is the net number of particles that pass from A to B?

3. A monatomic ideal gas initially occupies a volume V at a temperature T. It then is *i)* isothermally compressed to volume $V/2$, and *ii)* adiabatically expanded back to volume V.

   (a) What is the final temperature of the gas?

   (b) What is the total change in entropy of the gas?