Qualifying Examination — January 2009

General Instructions

- No reference materials are allowed.

- Do all your work in the corresponding answer booklet.

- On the cover of each answer booklet put only your *assigned number* and the part number/subject.

- Turn in the questions for each part with the answer booklet.

- 90 minutes are allotted for each part, except for Thermal Physics (45 minutes).

- Calculator policy: Use of a graphing or scientific calculator is permitted provided it lacks ALL of the following capabilities: (a) programmable, (b) algebraic operations, and (c) storage of ascii data. Handheld computers, PDAs, and cell phones are specifically prohibited.

Part I: Electricity and Magnetism

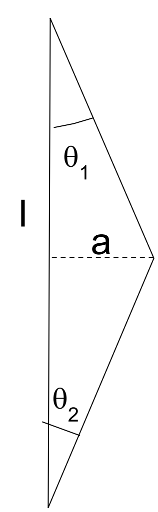
Do any 5 of the 6 problems

1.) Consider a region of space in which the magnetic field is zero at time $t = 0$ and there is no charge or current. The initial electric field is given by

$$\mathbf{E}(x, y, z) = ax^2\hat{y}$$

where $\hat{y}$ is a unit vector in the y direction.
a) Calculate the initial value of $\partial\mathbf{B}/\partial t$ everywhere.
b) Assuming that $\partial\mathbf{B}/\partial t$ is independent of time, calculate the magnetic field at all later times, $\mathbf{B}(x, y, z, t)$.
c) On a sketch of the x axis, draw the electric and magnetic fields at at least 5 points along this axis at $t = 0$.
d) Assuming your $\mathbf{B}$ is correct, calculate the electric field at all positions and times.
e) Check whether the fields you have calculated satisfy Maxwell's equations at all times. Was the assumption you made in part (b) correct?
f) On another sketch of the x axis, draw the electric and magnetic fields at at least 5 points along this axis at $t > 0$.

2.) a) Compute the magnetostatic field at the center of a wire loop of radius a, in terms of a, μ_0, and the current I.
b) Would you expect the field to be larger or smaller if the loop was the circumscribed square (that is, a square of side $2a$ that just touches the circle at 4 points)?
c) Compute the ratio of these two fields (algebraically and numerically) – was your expectation correct?
Hint: the field of a straight wire segment as shown in the figure below, at a distance a, is proportional to $cos\theta_1 + cos\theta_2$. Can you deduce the proportionality factor?



3.) A long coaxial cable has an inner conductor of radius a and an outer conductor of radius b, with charge per unit length λ and $-\lambda$ respectively.

a) Compute the electric field as a function of the distance r from the axis.
b) Compute the potential difference between the two conductors
c) Compute the capacitance of a length L of this cable.

4.) Consider a circuit consisting of an inductor of inductance L and a capacitor of capacitance C, in series.

a) Write the voltage across the capacitor in terms of its instantaneous charge $Q(t)$.
b) Write the voltage across the inductor in terms of $Q(t)$.
c) Write a differential equation for $Q(t)$.
d) Give a formula for $Q(t)$ satisfying the initial condition $Q(0) = Q_0$, with zero initial current. Define any parameters in terms of C and L.

5.) a) Write the two Maxwell equations for $\partial\mathbf{B}/\partial t$ and $\partial\mathbf{E}/\partial t$.
b) Assume no currents and sinusoidal $\mathbf{E}(r,t) = \text{Re}\,\mathbf{E}_0 \exp(\mathbf{k}\cdot\mathbf{r} - i\omega t)$ and $\mathbf{B}(r,t) = \text{Re}\,\mathbf{B}_0 \exp(\mathbf{k}\cdot\mathbf{r} - i\omega t)$, with some real wave vector $\mathbf{k}$. Take the the z axis along $\mathbf{k}$, and the y axis along $\mathbf{E}_0$, assumed real. Deduce a relation between ω and $\mathbf{k}$.
c) Write the complex amplitude $\mathbf{B}_0$ in terms of $\mathbf{E}_0$ and $\hat{\mathbf{k}}$, a unit vector along $\mathbf{k}$.

6.) Two positive charges, $q_1 = 1\ \mu\text{C}$ and $q_2 = 2\ \mu\text{C}$ lie on the x axis at $x = 0$ and $x = 2$ cm respectively.

a) Are there points at which a third charge would have no net force on it? If so, find all such points.
b) Are there points where the potential vanishes? If so, find all such points.

Part II: Quantum Mechanics

Do any 5 of the 6 problems

1.) A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = \infty\ , \qquad x < 0$$

$$U(x) = kx\ , \qquad x > 0$$

where k is a positive constant.
a) Of the following functions, choose the one which would best approximate the ground state eigenfunction in the interval $x > 0$. For each of the ones you do not choose, explain why. (A, b, and c are positive constants.)

$$\psi(x) = Ae^{-bx}$$

$$\psi(x) = Ax^2 e^{-bx}$$

$$\psi(x) = A \tanh bx$$

$$\psi(x) = Ae^{-bx} \sin cx$$

b) Use the variational principle to estimate the ground state energy of this particle. To simplify the math, use $\psi(x) = Axe^{-bx}$ as the trial wave function for $x > 0$. Give your answer in terms of m, $\hbar$ and k. Note that

$$\int_0^\infty u^n e^{-u} du = n!$$

2.) The Hamiltonian for a spin-1 system is given by

$$H = \alpha \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 2i \\ 0 & B & 1 \end{pmatrix}$$

where α is a constant.
a) Why must B be $-2i$?
b) What are the possible outcomes of a measurement of the system's energy?
c) At time $t = 0$, the state of the particle is described by

$$\phi = a \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

where a is a constant. What is the probability that a measurement of the system's energy at that time will *not* yield the result 3α?

3.) A particle is trapped in an infinitely deep, one dimensional square well, located at $0 \leq x \leq L$. Show that the energy eigenfunctions $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ satisfy the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$, where Δ denotes the rms variation.

Useful integrals

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4}\sin(2x) - \frac{1}{8}\cos(2x)$$

$$\int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \frac{x}{4}\cos(2x) - \frac{1}{8}\left(2x^2 - 1\right)\sin(2x)$$

4). A particle of mass m is trapped in a one dimensional oscillator potential, $U(x) = \frac{1}{2}m\omega^2 x^2$, for $-\infty < x < \infty$. At the classical turning points, $x = \pm A_0$, the kinetic energy of the particle is zero.

a) Find the location of the turning points associated with the nth energy eigenstate.

b) For the oscillator in the ground state calculate the probability of finding the particle beyond the classical turning points ($|x| > A_0$).

c) Assume the particle is trapped in a "half oscillator potential" of the following form: $U(x) = \infty$ for $x < 0$ and $U(x) = \frac{1}{2}m\omega^2 x^2$ for $x \geq 0$. Give the ground state energy and wave function for this case.

Harmonic oscillator energy eigenfunctions:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_2(x) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{m\omega}{\hbar}\left(2x^2 - \frac{\hbar}{m\omega}\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_3(x) = \frac{1}{\sqrt{3}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{m\omega}{\hbar}\right)^{3/2}\left(2x^3 - 3\frac{\hbar}{m\omega}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\int_{\sqrt{2}}^{\infty} e^{-\frac{x^2}{2}} \, dx \approx 0.072$$

5). Using a hydrogen sample an experimentalist determined the energy, orbital angular momentum, and z-component of the orbital angular momentum of the electron. The energy was found to be $E \approx -1.51\ eV$. What would be the possible outcomes for the other observables and what are their degeneracies? Your answer should indicate correlations between the different observables.

6). A beam, composed of equal parts of spin-up and spin-down particles of mass m, is moving to the right and is free of any external forces, $U = 0$ for $x < 0$. The beam is monochromatic and the particles have energy E. At $x = 0$ the particles encounter an infinitely wide potential barrier of height U_0 such that $U = U_0 > 0$ for spin-up particles and $U = -U_0$ for spin-down particles, for $x \geq 0$. Let $E < U_0$. For $x < 0$, $U = 0$, while for $x \geq 0$ the potential may be expressed in matrix form as:

$$\begin{pmatrix} U_0 & 0 \\ 0 & -U_0 \end{pmatrix}$$

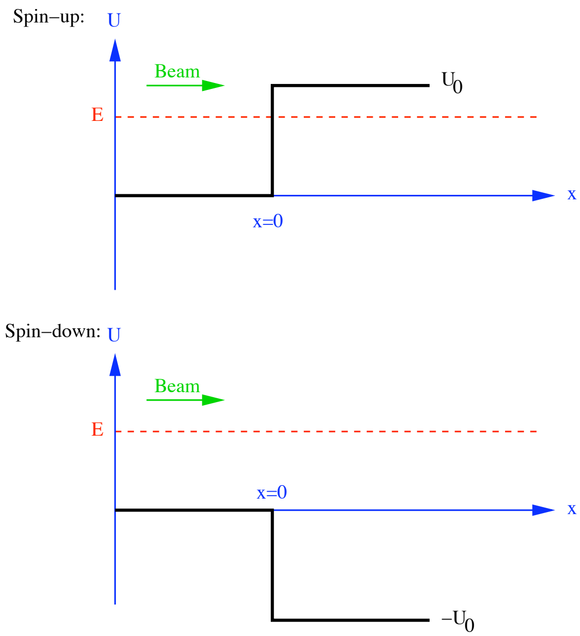


Figure 1: Particles scattering off a spin-dependent potential barrier.

a) Express the general form of the wave function in the regions $x < 0$ and $x \geq 0$ in terms of m, $\hbar$, E, U_0 and integration constants. Use spinor notation.
b) Take the amplitude of the incident right moving wave to be one. Calculate the remaining amplitudes from the boundary conditions and write down the resulting wave function.

Part IV: Thermal Physics

Do any 2 of the 3 problems

1. Two identical classical ideal gases with equal temperatures T and equal numbers of particles N, but with different pressures P_1 and P_2 (and volumes V_1 and V_2) are contained in separate vessels. Find the change in entropy when the vessels are connected and the gas comes to thermal equilibrium at temperature T.

2. (a) One mole of an ideal gas expands from a volume $V_1 = 3 \times 10^{-3}$ m^3 to a volume $V_2 = 10 \times 10^{-3}$ m^3 at a *constant temperature* of $T = 0$ °C. How much energy transfer by heat occurs with the surroundings during this process?
(b) Now the gas is returned to its the original volume at *constant pressure*, how much work is done on the gas during this phase?
(c) The system returns to its initial state by a *constant volume* process. What is the efficiency of this engine and how does it compare with that of a Carnot engine having the same minimum and maximum temperatures?

3. 1 kg of water at 0 °C is mixed with an equal mass of water at 100 °C. What is the change in entropy of the system after it has reached equilibrium? Assume that no energy is lost to the surroundings and that the specific heat of water $c_W = 4186 \frac{\text{J}}{\text{kg °K}}$ is constant over the relevant temperature range.

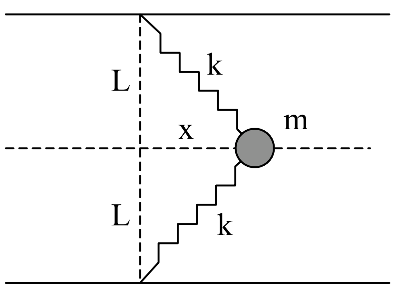
$$R = 8.314 \, \frac{\text{J}}{\text{mole °K}}$$

PART III: Classical Mechanics

Do any 5 of the 6 problems

1) A block of mass m is pulled along a flat surface by a rope inclined above the surface at an angle θ. The coefficient of kinetic friction between the block and the surface is μ. What angle of pull requires the least force to move the block at constant speed ? What is the minimum force?

2) A particle of mass m is attached to two identical (massless) springs of force constant k on a frictionless table as shown in the figure. One end of each spring is attached to the table. In the equilibrium position the springs are collinear ($x = 0$).



(a) Find an expression for the force on the particle as a function of the distance x from equilibrium.

(b) If the mass is displaced a distance $x = A$ and released from rest, what is the maximum speed of the mass?

3) (a) Derive an expression for the period of small oscillations of a physical pendulum.

(b) What is the period of oscillation of a meter stick about a pivot through the 20 cm mark?

4) A block of mass m sliding along a horizontal surface suffers a viscous resistance that varies with speed as $F(v) = -c\sqrt{v}$. (The minus sign indicates that the force is in the opposite direction of the velocity.) The initial speed of the block is v_0.

(a) Derive an expression for the speed as a function of position, $v(x)$, in terms of c, m, and v_0.

(b) How far does the block slide before coming to rest?

5) A 50-kg child stands beside a circular platform rotating at 0.1 rev/s. The platform is in the shape of a uniform disk and has radius of 2 m and a mass of 200 kg. The child steps onto the edge of the platform.

(a) What is the final rotational speed of the platform?

(b) How much mechanical energy, if any, is lost?

(c) What is the impulse exerted on the child by the platform pivot when the child steps on it?

6) Consider a spherical pendulum (a simple pendulum that can move in 3D).

(a) Use Lagrange's equations to determine the equations of motion.

(b) Show that the angular momentum corresponding to the azimuthal motion is constant.

(c) Show that the equations of motion revert to those of a simple pendulum when the motion is constrained to a vertical plane.