# Qualifying Examination – January 2007

## General Instructions

- No reference materials are allowed (except for the use of a calculator).
- Do all your work in the corresponding answer booklet.
- On the cover of each answer booklet put only your *assigned number* and the part number/subject.
- Turn in the questions for each part with the answer booklet.
- 90 minutes are alloted for each part, except for Thermal Physics (45 minutes).

## Part I: Electricity and Magnetism

*Do any 5 of the 6 problems*

1. Calculate the magnetic moment $\mathbf{m}$ of a disk of radius $R$ and thickness $T$ containing a volume charge density $\rho$ and rotating with a constant angular speed $\omega$ about the symmetry axis $z$.

2. (a) Write down Maxwell's equations;

   (b) Which equation suggests the introduction of the magnetic vector potential $\mathbf{A}$? State the mathematical and physical justification for this.

   (c) In free space find the wave equation satisfied by either $\mathbf{E}$ or $\mathbf{B}$.

   Given:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}, \quad \forall \text{ vector } \mathbf{v}.$$

3. Consider a spherical dielectric having a permittivity $\varepsilon$ and radius $R$. The sphere has a polarization $\mathbf{P}$ of the form

$$\mathbf{P} = kr^2 \, \hat{\mathbf{r}},$$

   where $k$ is a dimensional constant.

   (a) Find the electric field $\mathbf{E}$ inside and outside the sphere;

   (b) Find the energy stored in the space outside the sphere.

4. A steady current $I$ flows down a long cylindrical wire of radius $R$. The current density $\mathbf{J}$ is distributed in such a way that the magnitude of $\mathbf{J}$ is proportional to the distance $s$ from the axis.

   (a) Find the proportionality constant in terms of $I$ and $R$;

   (b) Find the magnetic field $\mathbf{B}$ inside and outside the wire.

5. An inverted hemispherical bowl of radius $R$ carries an uniform charge density $\sigma$. Find the potential difference between the "North pole" and the center, i.e., on the symmetry axis at the equator.

6. Consider a pair of parallel plates each having an area $A$, separated by a distance $d$. There is a constant potential difference $V$ between the plates. A dielectric having a dielectric constant $K = \varepsilon/\varepsilon_0 = 2$ fills half the volume between the plates. Find the capacitance for the following configurations:

   (a) the dielectric is parallel to the plates;

   (b) the dielectric fills only the left 1/2 of the volume.

## Part II: Quantum Mechanics

*Do any 5 of the 6 problems*

1. Which two quantities can be measured simultaneously in quantum mechanics?

   (a) position and momentum;
   (b) momentum and kinetic energy;
   (c) momentum and potential energy;
   (d) position and kinetic energy;
   (e) kinetic energy and potential energy.

   Say the angular momentum operator in the $x$-direction, $\hat{L}_x$, commutes with the Hamilton operator, $\hat{H}$. Which statement does NOT follow?

   (a) The angular momentum in the $x$-direction and the energy can be simultaneously measured;
   (b) The dynamical system is invariant under rotations about the $x$-axis;
   (c) The expectation value of $\hat{L}_x$ is conserved;
   (d) $\hat{L}_x$ and $\hat{H}$ have discrete eigenvalues;
   (e) $\hat{L}_x$ and $\hat{H}$ share the same eigenvectors.

   Given that $\hat{x}$ and $\hat{p}$ are hermitean operators, which of the following are NOT hermitean operators?

   (a) $\hat{x}^2$ (b) $\hat{p}^2$ (c) $\hat{p}\hat{x}\hat{p}$ (d) $[\hat{x}^2, \hat{p}^2]$ (e) $\hat{x}\hat{p} + \hat{p}\hat{x}$.

   Given that $\hat{A}$ is a hermitean operator, which statement is NOT true?

   (a) $< f|\hat{A}g >=< \hat{A}f|g >$, for any vectors $|f>$ and $|g>$ in the Hilbert space;
   (b) $\hat{A}$ equals its hermitean conjugate;
   (c) All of $\hat{A}$'s eigenvalues must be positive;
   (d) The eigenvectors of $\hat{A}$ are orthogonal;
   (e) The eigenvectors of $\hat{A}$ form a basis for the Hilbert space.

2. Which answer is FALSE for stationary states?

   (a) They must be independent of time (assume non-zero energy);
   (b) They must be energy eigenstates;
   (c) Expectation values evaluated for stationary states are independent of time;
   (d) The associated probability densities are independent of time;
   (e) They solve the Schrödinger equation.

   The state vector for a spin one-half particle is

$$|\psi>=\frac{1}{\sqrt{4}}|+> \; + \; \alpha|->,$$

   where $|\pm>$ are states whose $\hat{S}_z$ eigenvalues are $\pm\hbar/2$. The constant $\alpha$ is (up to a phase)

   (a) 0 (b) $\frac{1}{\sqrt{4}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{\frac{3}{4}}$ (e) 1.

   The likelihood of obtaining the eigenvalue $\hbar/2$ in a measurement of $\hat{S}_z$ is

   (a) 0% (b) 25% (c) 50% (d) 75% (e) 100%.

   Say that the $\hbar/2$ is in fact obtained in a measurement of $\hat{S}_z$. What is the likelihood of obtaining $-\hbar/2$ in a second measurement of $\hat{S}_z$, immediately after the first measurement?

   (a) 0% (b) 25% (c) 50% (d) 75% (e) 100%.

3. A particle is in the ground state of a one-dimensional infinite well with walls at $x = 0$ and $x = a$. The wall at $x = a$ is suddenly moved to $x = 2a$. If the wave function doesn't change when the wall is moved, what is the likelihood that the energy of the particle doesn't change?
   Useful integral:
$$\int d\theta \sin^2\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta.$$

4. The wave function for a free particle in one dimension at time $t = 0$ is
$$\psi(x,0) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; \frac{e^{ikx}}{\sqrt{k_0^2 + k^2}}.$$
   (a) What is the probability that the particle momentum is between $-\hbar k_0$ and $\hbar k_0$?
   (b) Write down an expression for the wavefunction at a later time. (Assume no measurement is made).

   Useful integral:
$$\int \frac{d\kappa}{\kappa^2 + 1} = \tan^{-1}\kappa.$$

5. A particle moves in one-dimension and feels the potential energy
$$V(x) = \kappa(x^2 - \ell^2)^2 \qquad \kappa > 0.$$
   (a) Find the stable equilibrium(a) of the particle.
   (b) If the particle undergoes *small* oscillations about the stable equilibrium(a), what is an approximate value for the ground state energy? (Hint: Taylor expand.)

6. Say that at time $t = 0$ the wavefunction for a particle inside an infinite well of size $a$ is
$$\psi(x,0) = \frac{\sqrt{2}}{5\sqrt{a}}\left(3\sin\frac{\pi x}{a} + 4\sin\frac{2\pi x}{a}\right) \qquad 0 \le x \le a,$$
   while it vanishes elsewhere. Show that the particle's most likely position oscillates inside the square well. Find the amplitude and period of the oscillation.
   Useful integral:
$$\int_0^{\pi} d\theta \; \theta \sin\theta \, \sin 2\theta = -\frac{8}{9}.$$

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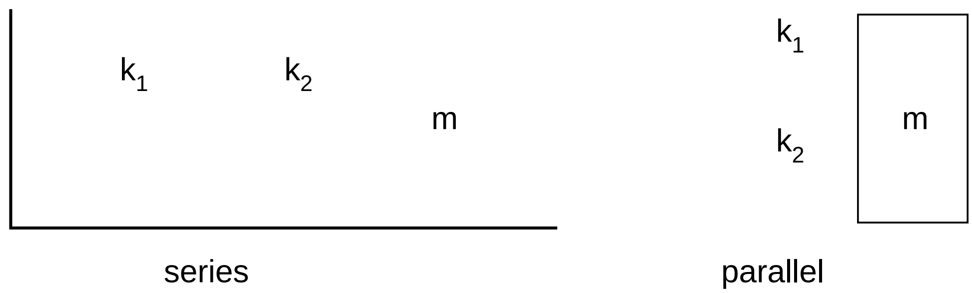
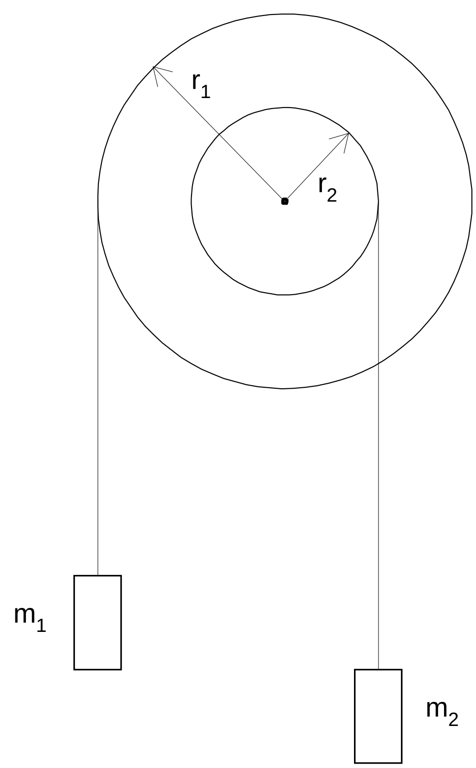
## Part III: Classical Mechanics

*Do any 5 of the 6 problems*

1. A mass $m$ is attached by two springs of constants $k_1$ and $k_2$ to a fixed vertical support – see figure below. Assuming no friction between the mass and the horizontal surface, calculate the effective spring constants and natural frequencies for the "series" and "parallel" configurations illustrated.



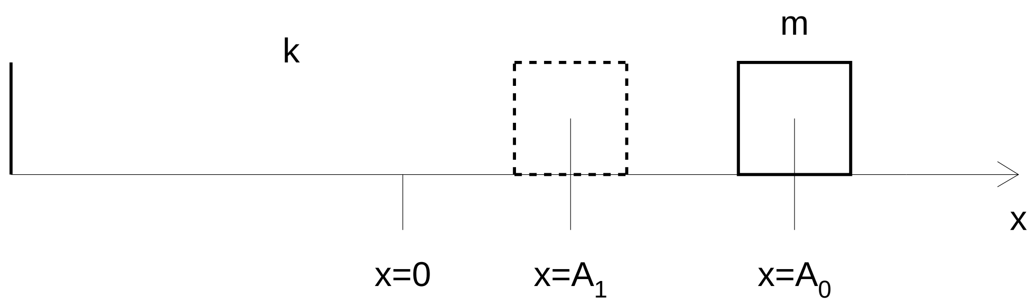

2. Calculate the acceleration of the masses $m_1$ and $m_2$ and the tensions in the strings ($T_1$ and $T_2$) for the modified Atwood machine shown in the figure below. The pulley has a moment of inertia $I$ and rotates clockwise.



3. A mass $m$ is attached by a spring of constant $k$ to a fixed vertical support and is pulled away from the position of equilibrium (unstretched spring) $x = 0$ by a distance $A_0$ – see the figure below. Assuming a coefficient of friction $\mu$ between the object and the horizontal surface, calculate

   (a) the amplitude $A_1$ after a full oscillation cycle;

   (b) the time for a full oscillation cycle.




4. (a) Calculate the period of a free falling object of mass $m$ through a tunnel going through the center of the Earth, starting from the surface, and assuming a constant-density Earth;

   (b) Calculate the period of an object of mass $m$ in a circular orbit around the Earth, at a height $h$ above the surface which is negligible with respect to the radius of the Earth (i.e., effectively $h = 0$);

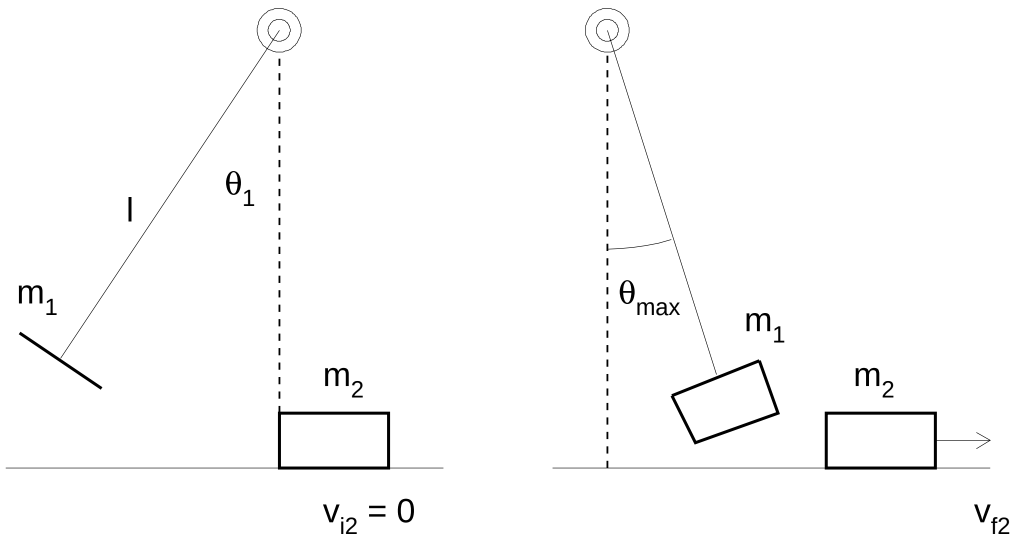
   (c) Calculate the free falling period of part (a) assuming that the entire mass of the Earth is concentrated in a small volume around the center (ignore relativistic effects).

5. A mallet of mass $m_1$ and length $\ell$ is used to strike a block of mass $m_2$, initially at rest on a horizontal, frictionless surface. The entire mass of the mallet is concentrated at the end and it is free to pivot about the opposite end. The mallet is raised to an initial angle $\theta_i$ and released with no initial velocity to strike the block $m_2$ when $\theta = 0$ (see figure).

   (a) Calculate the speed of the mallet at the instant before it strikes the block;

   (b) Calculate the final speed of the block $m_2$, $v_{f2}$, assuming that the collision is perfectly elastic;

   (c) Calculate the maximum angle, $\theta_{max}$, the mallet reaches after striking the block.




6. Consider two pointlike particles of masses $m_1$ and $m_2$, at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, attracted by a force which depends on their separation $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The potential energy then depends on the distance $r$, $U = U(r)$.

   (a) Express the Langrangian of this system as a function of $\mathbf{r}_1$ and $\mathbf{r}_2$.

   (b) Rewrite this expression as a function of the position of the center of mass, $\mathbf{R}$, and the particle separation $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

   (c) Derive the equation of motion for the center of mass of the system and describe the motion (e.g., what is the acceleration and the velocity).

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## Part IV: Thermal Physics

*Do any 2 of the 3 problems*

1. n mols of an ideal, monoatomic gas have a volume $V_i$ at pressure $P_i$ and temperature $T_i$. This gas is compressed to 1/2 of its original volume in three different ways, namely: (a) isobaric, (b) isothermal, (c) adiabatic.

   (i) Sketch in a $(P, V)$ diagram the three processes and label accordingly;

   (ii) Calculate the amount of work required for each process;

   (iii) State for each of the three processes whether the temperature increases, decreases, or remains constant;

   (iv) Calculate the final temperature for the isobaric process;

   (v) Calculate the final pressure for the isothermal process.

2. A room air conditioner operates as a Carnot cycle refrigerator between an outside air temperature, $T_h$, and an inside temperature, $T_i$. The room gains heat from outside at a rate of $A(T_h - T_i)$. This heat is removed by the air conditioner. Assuming that the power supplied to the cooling unit is $P$, show that the steady state temperature of the room is

$$T_i = T_h + P/2A - \sqrt{(T_h + P/2A)^2 - T_h^2}.$$

3. The largest bottle ever made by blowing glass has a volume of about $V_i = 0.720\,\mathrm{m}^3$. Imagine that this bottle is filled with air that behaves as an ideal diatomic gas. The bottle is held with its opening at the bottom and rapidly submerged into the ocean. No air escapes or mixes with water; no energy is exchanged with the ocean by heat.

   (a) If the final volume of the air is $V_f = 0.300\,\mathrm{m}^3$, by what factor does the internal energy of the air increase?

   (b) If the bottle is submerged so that the absolute air temperature triples, how much volume is occupied by air?

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