Name: ______________________ **Number:** ____________

QUALIFYING EXAMINATION

SPRING 2004

General Instructions: Your exam proctor will assign your qualifier ID number at the start of the test. This will be the same for Parts I through IV. No reference materials (other than a calculator) are permitted. Do all work in your answer booklet. Turn in your answer booklet. Turn in the questions for each part with the answer booklet. Put your name on the test as well as your assigned number. Ninety minutes are allowed for each part with a 30-minute break in between. You may finish Part I early, turn it in and start on Part II.

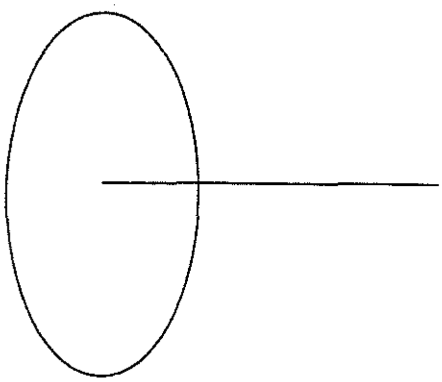
Only put your number on the front of the answer booklet. Do not put your name on the answer booklet. Put your name on the test as well as your assigned number.

PART I: Classical Mechanics

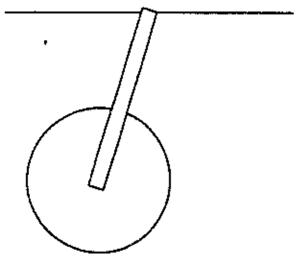
Work any 5 of the 6 problems.

1. A particle of mass m is released from rest on the axis of a circular ring of mass M and radius R at a distance x_0 from the center of the ring. The particle is constrained so that it can only move along the axis (without friction). Assume M >> m so that M remains fixed. (The particle moves in response to the gravitational pull of the ring).

   (a) What is the speed of m when it reaches the center of the ring?
   (b) If x << R, determine the time for the mass to reach the center of the ring.
   (c) Write down an integral expression for the time to reach the center of the ring if released at an arbitrary position. (You don't need to evaluate the integral.)



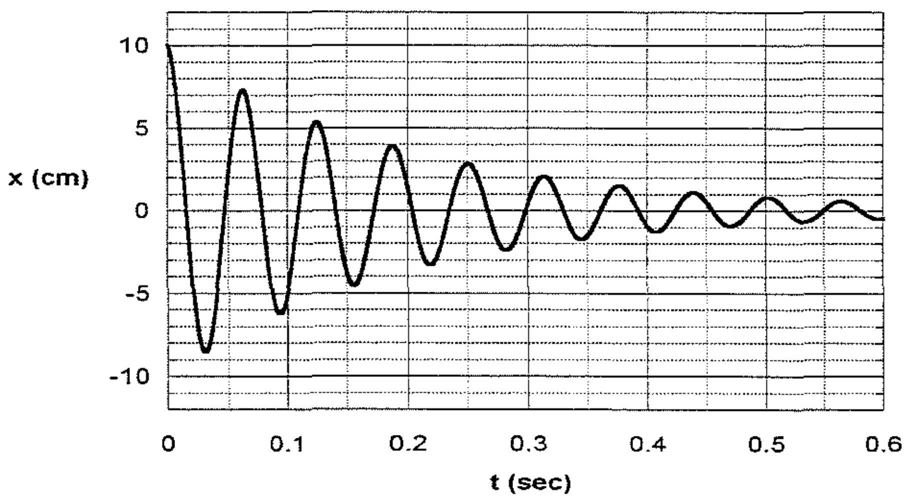
2. A pendulum consists of a light rod of length L = 1 m attached to the center of a disk of radius R and mass M = 0.5 kg. The pendulum undergoes small oscillations with the disk oriented in the plane of oscillation. If the period of oscillation is 2.27 sec, what is the radius of the disk?



3. A mass, *m*, at the end of a spring, with spring constant, k, is subject to a resistive force, in addition to the Hook's law force F=-kx, that varies linearly with velocity ($F_r = -cv$).

(a) Derive an expression for the position of the mass as a function of time valid for small damping. (Small means $c \ll (km)^{0.5}$).

(b) If the mass is displaced from equilibrium and released from rest, its position is found to vary with time as shown in the graph below. Assuming that m = 0.2 kg, determine the damping constant, c, and the force constant, k, of the spring.



4. Showing all steps, derive the expression for the rotational inertia of a solid sphere of uniform mass density about an axis through the center of the sphere.

5. An artillery shell is fired at an angle of elevation of 60° with initial speed v_0. All the energy goes into the kinetic energy of the fragments. At the uppermost part of its trajectory, the shell bursts into two equal fragments, one of which moves directly upward, relative to the ground, with initial speed $v_0/2$.

(a) What is the direction and speed of the other fragment immediately after the burst?
(b) How much energy was released when the shell exploded?

6. Write down the Lagrangian in polar coordinates for a mass moving on a plane subject to a central force, $f(r) = -dV/dr$.

(a) Show how conservation of angular momentum results from one of the equations of motion.
(b) From the equations of motion, obtain the differential equation for the radial motion of the mass, which is independent of the angular variable and its derivative.
(c) What is the effective potential describing the radial motion?
(d) Sketch this effective potential for the case of gravity and discuss how the shape of the effective potential relates to the stability of the orbit.

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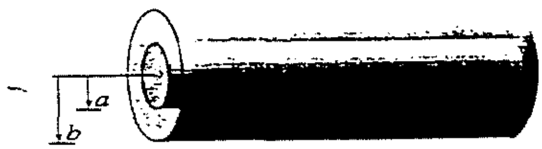
PART II: Electricity and Magnetism

Work any 5 out of 6 problems.

1. Find the capacitance per unit length of two hollow coaxial metal cylinders, of radius a and b. Assume the inner cylinder has +Q and the outer cylinder –Q.



2. Consider two vectors

$\mathbf{v}_1 = k_1[(x + 2y + 4z)\,\mathbf{i} + (2x - 3t - z)\,\mathbf{j} + (4x - y + 2z)\,\mathbf{k}]$

$\mathbf{v}_2 = k_2[(xy)\,\mathbf{i} + (2yz)\,\mathbf{j} + (3xz)\,\mathbf{k}]$

where k_1 and k_2 are dimensional constants, and **i, j,** and **k** are the unit Cartesian vectors. One of these represents an electric field **E** in the absence of any sources while the other represents a vector potential **A.**

(A) For the vector representing **E**, find the corresponding potential V (x, y, z).

(B) For the vector representing the vector potential **A**, find the corresponding magnetic field **B** (x, y, z).

3. A point charge q is a distance z above an infinite grounded conductor in the x-y plane.

(A) Find the force (magnitude and direction) on q.

(B) Find the dipole moment for the system.

4. An infinitely long wire of cross-sectional radius R has a current density $J = C \exp(ks^2)$, where C and k are dimensional constants, and s is the radial distance from the center of the wire.

(A) Find **B** for $s < R$.

(B) Find **B** for $s > R$.

5. A circular loop of wire of radius R and mass m "floats" vertically in the plane of the paper in a constant magnetic field **B** pointing out of the page. The magnetic field passes only through the upper half of the loop. Find the current I, magnitude and directions, necessary to maintain the loop in equilibrium with gravity toward the bottom of the page.

6. A sphere of radius R carries a charge density $\rho(r) = C\,r$, where C is a constant.

(A) Find the electric field inside the sphere and outside the sphere.

(B) Find the energy stored in the field.

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PART III: Quantum Mechanics

Work any 5 out of 6 problems.

1. (a) Write the quantum mechanical operator corresponding to the momentum **p** of a particle in the position representation.
   (b) Write the quantum mechanical operators for the three Cartesian components of **r X p** in the position representation.
   (c) Write relations between the Cartesian coordinates x, y, z and the cylindrical coordinates r, $ø$, z.
   (d) Explicitly compute $(\mathbf{r} \times \mathbf{p})_z$ in terms of ∂/∂, $\partial/\partial ø$, $\partial/\partial z$.
   (e) Write the eigenvalues, and the most general eigenfunctions, of $(\mathbf{r} \times \mathbf{p})_z$.

2. Two non-relativistic particles of equal mass m move in one dimension, tethered together (*i.e.*, attached to each other) by a string of length L, so that the potential $V(x_1, x_2) = 0$ if $|x_2 - x_1| < a$ and $V(x_1, x_2) = \infty$ if $|x_2 - x_1| \geq a$. Find the eigenfunctions and eigenvalues of the Hamiltonian of this system.

3. Consider a particle moving in the $+x$ direction incident on a downward potential step ($V = 0$ for $x < 0$, $V = -V_0$ for $x \geq 0$).
   (a) Compute the reflection probability in terms of the incident energy E and V_0.
   (b) Compute this probability numerically for the case $E = V_0$.

4. Use the Bohr-Sommerfeld quantization condition $\oint pdq = nh$ to determine which classical orbits of the harmonic oscillator with spring constant k and mass m are quantum mechanically allowed, and therefore what are the allowed energies. Compare to the exact quantum mechanical energies.

5. A particle of mass m is in the ground state of a one-dimensional well bounded by infinite potential barriers at $x = 0$ and $x = L$. Suddenly, the right hand wall is moved to $x = 2L$, so the particle is in an infinite square well of width $2L$. What is the probability that the particle is in the ground state of the new well?

6. Assume the angular momentum operators **L** satisfy the commutation relation $[\mathbf{L}_x, \mathbf{L}_y] = i\hbar\mathbf{L}_z$ and its cyclic permutations.
   (a) Define the raising operator $\mathbf{L}_+$ and verify that it raises the eigenvalue of $\mathbf{L}_z$ by ħ.
   (b) Compute the commutator $[L_+, L_-]$, where $L_- \equiv L^{\dagger}_+$.
   (c) If $|L, M>$ denotes the normalized simultaneous eigenvector of L^2 and L_z with eigenvalues $\hbar^2 L(L+1)$ and $M\hbar$, find the constant A such that $|L, L-1> = AL_- |L, L>$.

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PART IV: Mixed Topics

Do problems from 4 of the 5 sections. Astrophysics may be chosen only in place of electronics. Do 6 of the 8 problems with at least one from each of the 4 chosen sections. Use a different answer book for each lettered section.

A: Relativity

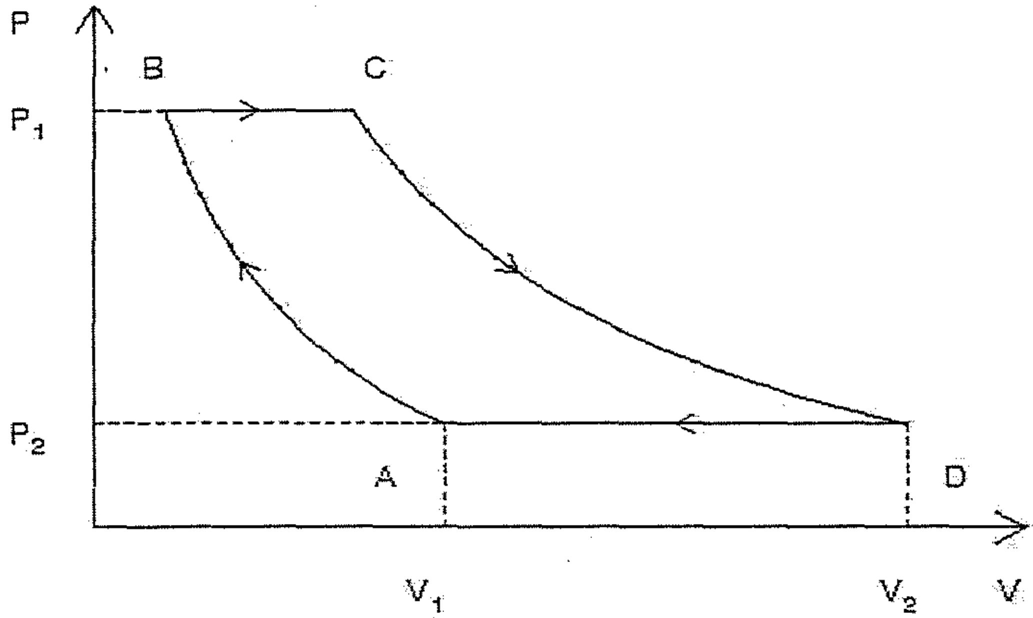
1. Inertial observer A detects events at (1.50 m, -1.75 m, 0.00 m. 0.00 ns) and (3.00 m, -1.75 m, 0.00 m, 2.00 ns), respectively, where the ordered quadruplets give the space-time coordinates (x, y, z, t). Inertial observer B detects the same two events to occur simultaneously. The respective y and z coordinates of the two events are the same for observer B as for observer A. (a) What is the velocity of observer B with respect to observer A? (b) What is the distance between the two events as measured by observer B? ns=nanosec

2. A proton is accelerated in vacuum by a constant force of magnitude 5.00×10^{-13} N. If the proton started from rest, what is its (a) momentum and (b) speed after 1.5 μs of acceleration?

   The rest mass of a proton is $1.673 \text{ X } 10^{-27}$ kg and the speed of light is $2.998 \text{ X } 10^{8}$ m/s.

B: Thermodynamics

1. An ideal gas is carried through a thermodynamic cycle consisting of two isobaric and two isothermic processes, as shown in the figure below (ABCA).



Show that the net work done in the entire cycle is

$$W = P_2 (V_2 - V_1) \ln \frac{P_1}{P_2}$$

2. (a) Show that the fraction of particles below an altitude h in the atmosphere is given by

$$f = 1 - \exp(-mgh/kT).$$

(b) Use this result to show that half of the particles are below the altitude $h' = kT\ln 2/mg$. What is the value of h' for the earth? (Assume a temperature of $T = 270\text{K}$ and an average molar mass for air of 28.9 g/mol.)

$$k = 1.38 \times 10^{-23} \text{ J/K}$$

C: Optics

1. A 1-inch tall candle is set 3 inches in front of a concave spherical mirror having a 12-inch radius of curvature. Describe the resulting image with a ray diagram and derive the magnification.

2. Consider the waves

$$\mathrm{E1}(z,t) = (E_{oz}\mathrm{i} - E_{oy}\mathrm{j})\cos(kz - wt)$$

$$\mathrm{E}_2(z,t) = (E_{ox}\mathrm{i} + E_{oy}\mathrm{j})\cos(kz - wt - \pi/2)$$

(a) Show that in general these waves are not orthogonal.
(b) Under what circumstances are the planes of vibration normal to each other? Determine the polarization state of the combined normal fields.

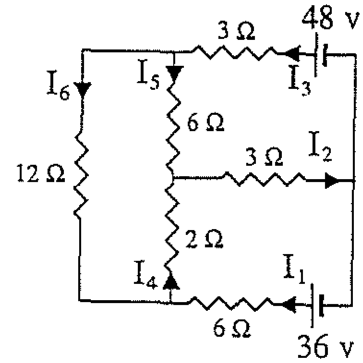
D: Astrophysics

1. A comet with radius R_c = 10 km and mass M_c = 10^{18} gm approaches Jupiter, which has mass M_J = 1.9 x 10^{30} gm and radius R_J = 7.1 x 10^4 km. If the comet is held together only by self-gravity, how close can it get to Jupiter before it is pulled apart by tidal forces? (Express in cm as well as a multiple of Jupiter's radii). At what distance from the Sun would the same comet be disrupted? The Sun has mass M = 2 x 10^{33} gm and radius R = 7 x 10^{10} cm. (Express both in terms of cm and solar radii.)

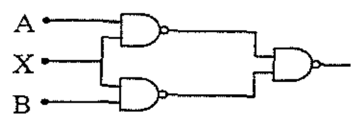
2. Imagine standing on the side of the Moon facing the Earth during the New Moon phase: the Moon's surface is dark where you are standing and the Earth appears fulls in the sky. How much brighter or fainter is the full Earth as seen from the Moon compared to the full Moon as seen from Earth? How much brighter or fainter would the lunar soil appear in the reflected light from the full Earth compared to the appearance of similar soil on Earth in the light of the full Moon? What is the implication for walking on the Earth at night in the light of the full Moon? The radius of the Earth is 6380 km, while that of the Moon is 1740 km. The mass of the Earth is 6.0 x 10^{27} gm, while that of the Moon is 7.4 x 10^{25} gm. The average albedo (reflectivity) of the Earth is about 0.3, while that of the Moon is about 0.07.

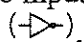
E: Electronics

1. A DC circuit is shown. It is known that the current passing through the 2Ω resistor, I_4, is 3 A and the current through the central 3Ω resistor, I_2, is 6 A. Determine the current through each of the other resistors as marked on the diagram.



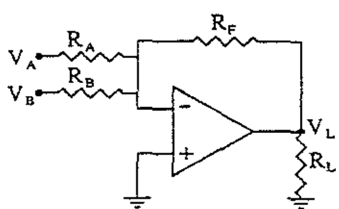
2. A. For the logic circuit shown, which consists entirely of nand gates, write down the truth table for inputs A and B with:
   (a) X held at logical zero (false).
   (b) X held at logical one (true).



B. Given several two input nand gates (), several two input nor gates (), and several inverters (), construct a logic circuit that performs an exclusive OR function: either A or B but not both A and B.

3. A. A simple operational amplifier circuit is shown in the figure. If $R_A = R_B = R_F$, determine how the output voltage, V_L, depends on the voltages of the two inputs, V_A and V_B.

Part B to this question is on next page.



B. A second operational amplifier circuit is shown in the figure. What is its function? Specifically, determine how the voltage V_L and the current through T_L depend on the value of T_L, assuming that T_f and V_b are fixed.