Qualifying Examination – January 2003

General Instructions

No reference materials are allowed (except for the use of a calculator). Do all your work in the answer booklet. On the cover of each answer booklet put only your *assigned number* and the part number. Turn in the questions for each part with the answer booklet. 90 minutes are alloted for each part, with a 30 minutes break in between.

Part I: Classical Mechanics

Do any 5 of the 6 problems

1. A particle of mass m moves on the surface of a cone with opening angle 2α and axis in the positive z–direction. Write down the particle Lagrangian and resulting equations of motion in terms of the distance from, and angle about, the z–axis. Include the gravitational force (in the negative z–direction).

2. In an elastic collision two particles A and B with equal kinetic energies, having masses m_A and m_B and speeds v_A and v_B, respectively, collide such that A is at rest after the collision. Find conditions relating v_A/v_B and m_A/m_B for this to be possible. (The collision is assumed to be 1–dimensional.)

3. A rod of length L has a mass per unit length that is proportional to the distance from one end. Determine the position of the center of mass.

4. A particle of mass m moves under the influence of the potential $V(x) = A/x^2 + Bx^2$. Determine the equilibrum position and the natural frequency of small oscillations about the equilibrium in terms of A and B.

5. How fast would the earth have to rotate, and how long would a day be, in order for the centrifugal force and gravity cancel at the equator? Use $R = 6.4 \times 10^6$ m.

6. Compute the tension in the string attached to a yo-yo as it is released vertically from rest. Assume the yo-yo has a mass M and moment of inertia I about its axis of rotation and that the string is wound around a spool of radius r.

Part II: Electricity and Magnetism

Do any 5 of the 6 problems

1. An elastic balloon has a mechanical potential energy for radius $r > R_0$ of

$$V_M(r) = V_0 \left(\frac{r - R_0}{R_0} \right).$$

If a charge Q is uniformly deposited on the balloon, what is its equilibrium radius?

2. A charge distribution, uniform in the z direction, depends on the distance r from the z axis as

$$\begin{aligned} \rho(r, \theta, z) &= \frac{Q}{R_0^2 r} \exp(-r/R_0) \qquad r < R_0 \\ \rho(r, \theta, z) &= 0 \qquad r > R_0 \end{aligned}$$

What is the electric field inside and outside the charge distribution?

3. In cylindrical coordinates (r, θ, z) the vector potential and current density are given by

$$\begin{aligned} \vec{A}(r, \theta, z) &= \hat{e}_\theta f(r) \\ \vec{j}(r, \theta, z) &= -\frac{15}{\mu_0 r^2} \vec{A} \quad \text{(SI units)} \quad \text{or} \quad \vec{j}(r, \theta, z) = -\frac{15c}{4\pi r^2} \vec{A} \quad \text{(Gaussian units)} \end{aligned}$$

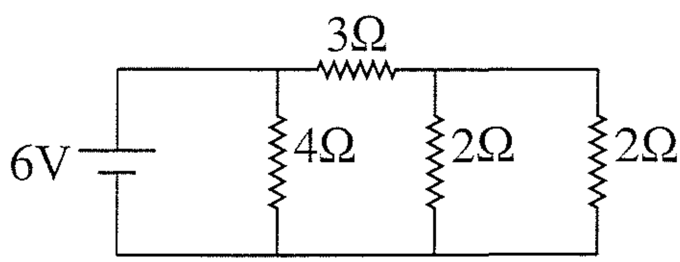
What is the functional form of $f(r)$ assuming it vanishes at $r = \infty$?
Note: you may choose either of the two unit systems. The Laplace operator reads:

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

4. At the center of a regular tetrahedron (whose surface is formed by four equilateral triangles of side length a) lies a point charge Q. What is the electric flux passing through each triangle?

5. A particle of mass 1.67×10^{-27} kg and charge 1.6×10^{-19} C traveling in the x direction with speed of 1 km/s enters a region of magnetic induction 1 T in the negative z direction. After 1 ns, in what direction is the particle traveling?

6. A six volt battery is attached to the circuit shown below. What current flows in each of the 2Ω resistors?



Part III: Quantum Mechanics

Do any 5 of the 6 problems

1. (a) What is the de Broglie wavelength of a 1 MeV electron?
   (b) A 1 MeV photon?
   Given: $h = 6.6 \times 10^{-34}$ J $\cdot$ s and the electron rest mass $m = 511$ keV.

2. Usė the Bohr quantization rules to calculate the energy levels for a harmonic oscillator for which the energy is

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2.$$

   Consider only circular orbits. Also obtain an expression for the inverse of the wavelengths of the photons which are emitted when the system goes from higher to lower energy states.

3. Consider the wave function

$$\psi(x,t) = A\, \mathrm{e}^{-\lambda\, |x|}\, \mathrm{e}^{-i\omega t}.$$

   (a) Normalize $\psi(x,t)$.
   (b) Determine the expectation values of x and x^2.
   (c) Find the standard deviation, σ, of x. Sketch the graph of $|\psi|^2$ and mark the points $(< x > +\sigma)$ and $(< x > -\sigma)$. What is the probability that the particle will be found outside of this range?

4. A particle in an infinite square well (extending from $x = 0$ to $x = a$) has a wave function

$$\psi(x,0) = A\, \left(\psi_1(x) + \psi_2(x)\right),$$

   where the subscripts label the energy levels for an infinite square well.

   (a) Normalize $\psi(x,0)$.
   (b) Find $\psi(x,t)$ and $|\psi(x,t)|^2$.
   (c) Compute the expectation value of the position, $< x >$, as a function of time.
   (d) Compute the expectation value of the momentum, $< p >$, as a function of time.
   (e) Show that the expectation value of the energy, $< E >$, is the average of the energies E_1 and E_2.

5. Suppose that five electrons are placed in a one-dimensional infinite potential well of length L. What is the energy of the ground state of this system of five electrons? What is thė net spin of the ground state? Take the exclusion principle into account, and ignore the Coulomb interaction of the electrons with each other.

6. Suppose that the electron in a hydrogen atom is perturbed by a repulsive potential concentrated at the origin. Assume the potential has the form of a 3-dimensional delta function, so the perturbed Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} + A\delta^3(\mathbf{r}),$$

where A is a constant. To first order in A, find:

(a) the change in the energy of the state with quantum numbers $n = 1, l = 0$. Hint:

$$\psi_{100}(\mathbf{r}) = \frac{2\exp(-r/a_0)}{\sqrt{4\pi}\, a_0^{3/2}}.$$

(b) the change in the wave function.

Part IV: Mixed Topics

Do problems from 4 of the 5 subsections (choose from either the astrophysics subsection or the electronics subsection). Do 6 of the 8 problems of these subsections, with at least one problem in each of the 4 subsections.

Use a different answer book for each subsection.

A. Special Relativity

1. A hypotetical particle which has a decay time $\tau = 10^{-9}$ s is created within a detector and is moving with a speed $v = 0.99\,c$.

   (a) What is the apparent mass of this particle with respect to its rest mass?

   (b) How far does this particle move before decaying?

2. An object moves with speed u for time Δt at an angle θ with respect to the line of sight (x-axis) to a stationary observer.

   (a) What is the time interval Δt^{obs} that the object appears to take to travel a distance x towards the observer as seen by the observer? Write Δt^{obs} in terms of u, c, θ and Δt.

   (b) What is the apparent transverse velocity of the object, v_y, seen by the observer? Write v_y in terms of u, c and θ.

   (c) Show that the transverse velocity is a maximum when $\cos\theta = u/c$. What is the maximum in terms of u, c and/or the Lorentz factor?

B. Thermal Physics

1. A system consisting of n *mols* of an ideal gas undergoes a reversible isobaric process from a volume V_i to a volume $3 \cdot V_i$. Calculate the change in entropy of the gas.

2. Use the Maxwell-Boltzmann *energy* distribution,

$$f(E)\,dE = \frac{2}{\sqrt{\pi}}(k_B T)^{-3/2}\sqrt{E}\,e^{-E/k_B T}\,dE,$$

and derive

   (a) The *most likely* kinetic energy E_{ML} of the molecules.

   (b) The *average* kinetic energy $\langle E \rangle$ of the molecules.

Note:

$$\Gamma(x) = \int_0^\infty t^{x-1}\,e^{-t}\,dt$$

The Γ-function has an interesting feature:

$$\Gamma(x) = (x-1)\Gamma(x-1),$$

i.e., it is the extension of the factorial to complex and real arguments. We also note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and that the energy distribution is (obviously) normalized:

$$\int_0^\infty f(E)\,dE = 1.$$

C. Optics

1. An object is at a depth d below the surface of a transparent material with refractive index n. As viewed from directly above (in air), how deep does the object appear to be?

2. White light is incident on a grating ruled with 2500 lines/cm. Calculate the angular separation between violet light ($\lambda = 400$ nm) and red light ($\lambda = 700$ nm) in

   (a) the first order;

   (b) the second order.

   (c) Does yellow ($\lambda = 600$ nm) in the third order overlap violet in the fourth order?

D. Astrophysics

1. We want to detect a planet of 1 Jupiter mass (adequately taken as 0.001 solar mass) in an edge-on circular orbit around a solar-type star, using the Doppler technique in the spectral region near the Mg and Fe lines around 5200 Angstroms. If this planet is in an orbit with a radius of 1 AU, what wavelength precision do we need to detect the star's radial-velocity variations at the 3σ level? State any assumptions you have to incorporate.

2. Recent observations have succeeded in maping the entire orbit of a star around an "object" near the galactic center. The elliptical orbit, 90° to the line of sight, has a period of 15 years, and ranges from 17 to 240 light-hours from the central mass. Estimate the magnitude of the central mass from these data, and show why the evidence is compelling that this "object" is a black hole.

E. Electronics

1. (a) Draw a circuit for a full wave power supply using a transformer, 2 diodes and a capacitor filter.

   (b) If the load resistor is 50 Ω and the input AC frequency is 60 Hz, what value of capacitor will give an AC ripple of less than 10%?

2. Draw a single RC circuit that will give the following outputs for the given inputs: