Qualifying Examination
January 2002

General Instructions: No reference material (except for the use of a calculator) are permitted. Do all your work in the answer booklet. On the cover of each answer booklet put only your assigned number and the part number. Turn in the questions for each part with the answer booklet. 90 minutes are allotted for each part, with a 30 minute break in between.

Part I — Classical Mechanics

Do any 5 of the 6 problems.

1. Determine which of the following forces are conservative. If the force is conservative, then determine the potential energy function.

$$\text{(a)} \quad \mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$$

$$\text{(b)} \quad \mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^3\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the x, y, z-directions, respectively.

2. A circular hoop of radius 20 cm hangs over a horizontal peg. Write down the appropriate Lagrangian function. What is the period of oscillation of the hoop if it is displaced slightly and allowed to oscillate? (Assume the hoop doesn't slip.)

3. A particle of mass $m_1 = 1$ kg with an initial speed $v_1 = 4$ m/s collides with a second mass $m_2 = 2$ kg initially at rest. After the collision, m_1 has a speed of $v_1' = 3$ m/s and travels at an angle $\theta_1 = 30^o$ from its initial direction. (a) Find the speed (v_2') and direction of travel (θ_2) of m_2 after the collision. (b) Find the loss of kinetic energy during the collision (if any).

4. The interatomic potential $V(r)$ of a diatomic molecule is given by

$$V(r) = V_0 \left(1 - e^{-(r-r_0)/\delta}\right)^2 - V_0 \,,$$

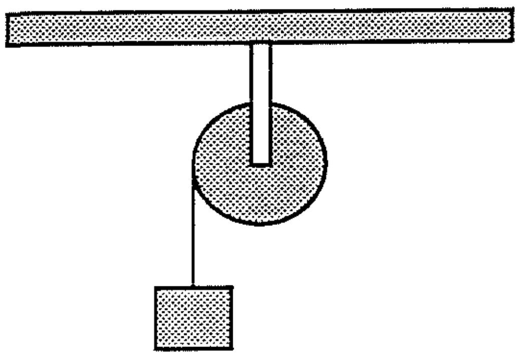
where r is the distance of separation. (a) What is the equilibrium separation of the atoms? (b) Derive an expression for the vibrational frequency, assuming small amplitudes, and that both atoms have the same mass m.

5. An object projected along a horizontal surface with an initial speed v_0 is subject to a retarding force given by

$$F(v) = -Ae^{\alpha v} ,$$

where A and α are constants. (a) Find its speed as a function of time. (b) Find the time for it to come to rest.

6. A 2-kg block hangs from the end of a light rope that is wrapped around a wheel as shown below. The wheel is pivoted so that it can rotate about its center. Find the speed of the block after it is released from rest and allowed to fall 1 m. The rotational inertia of the wheel is 0.1 kg m^2 and its radius is 0.4 m. Neglect friction.



Do any 5 *of the* 6 *problems.*

Constants: $\varepsilon_0 = 8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{Nm}^2$ $\quad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A}^2$ $\quad c = 3.0 \times 10^8\ \mathrm{m/s}$

1. A point charge of 1 μC (microcoulomb) lies at the origin, and another charge -2 μC lies along the x axis at $x = -1$ m.
   (a) What is the force on the charge at the origin?
   (b) What is the electric field at the point $x = +1$ m (due to both charges)?
   (c) Where on the x axis is the electric field equal to zero?

2. (a) Write the Maxwell equations giving $\frac{d\mathbf{E}}{dt}$ and $\frac{d\mathbf{B}}{dt}$ in a vacuum, in any system of units. (b) Show how these equations can be converted to a wave equation of the form
$$\frac{d^2\mathbf{E}}{dt^2} = v^2 \nabla^2 \mathbf{E}$$
and give a formula for the wave speed v in terms of the constants in the Maxwell equations.

3. In a region of space, the electric field at $t = 0$ is known to be given by $E_x(\mathbf{r}, t) = Nx$, $E_y(\mathbf{r}, t) = Ny$, and $E_z(\mathbf{r}, t) = Nx$ (*not* Nz), for some constant N. The magnetic field is initially zero, and there is no current at any time.
   (a) Compute the charge density ρ at time $t = 0$.
   (b) Compute the rate of change of the magnetic field vector.
   (c) Give the electric and magnetic fields at an arbitrary time t.

4. Consider two slabs of dielectric material with dielectric constants 2.0 and 3.0. Each slab is 1 cm thick and they are placed between two conducting plates 2 cm apart. What is the capacitance per unit area (in units farads/m^2) of this arrangement? Give the values of the D and E fields in both slabs for the case that the voltage difference between the conducting plates is 100 volts.

5. A point charge q lies at the center of a cube of side a. What is the electric flux passing out each of the square faces of the cube?

6. A long cylindrical pipe of inner radius R has many axial wires glued to the inside, held at different potentials such that the potential at radius R is $V(R, \theta, z) = V_0 \cos(6\theta)$.

(a) Compute the potential $V(r, \theta, z)$ at an arbitrary radius $r < R$.

(b) Compute the surface charge density (charge per unit area) $\sigma(\theta, z)$ on the system of wires.

Do any 5 of the 6 problems.

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. A particle of mass m is confined to a one–dimensional box of width L, that is, the potential energy of the particle is infinite everywhere except in the interval $0 < x < L$, where its potential energy is zero. The particle is in its ground state. What is the probability that a measurement of the particle's position will yield a result in the left quarter of the box?

2. A spin–$\frac{1}{2}$ particle is subject to the interaction potential
$$U = \lambda \mathbf{S} \cdot \mathbf{B}$$
where λ is a constant and $\mathbf{B} = B_0 \mathbf{j}$. ($\mathbf{j}$ is the unit vector in the y-direction.) Find the eigenvalues and eigenstates of U (expressed in two–component spinor notation) in terms of λ, B_0, and physical and mathematical constants. Assume the representation $S_i = \hbar\sigma_i$ for the spin operators.

3. Suppose that the energy and *total* angular momentum of a hydrogen atom electron are measured and that the outcome of the energy measurement is -3.4 eV. [*Hint:* The ground state energy is -13.6 eV.] In units of $\hbar$, what would the possible outcomes for the total angular momentum measurement have been? Explain.

4. A one–dimensional system is in a state for which the expectation values of the position and momentum operators x and p are zero, and the expectation value of x^2 is 0.01 nm^2. What would be the minimum possible value for the expectation value of p^2 in units of kg^{2}m^2/s^2 ?

5. Consider a particle moving in one dimension under the influence of the following potential:
$$V(x) = \begin{cases} \infty & x < -b \\ 0 & -b < x < -a \\ V_0 & -a < x < a \\ 0 & a < x < b \\ \infty & x > b \end{cases} \tag{1}$$

Assume that $b \gg a$. Use first–order perturbation theory to estimate the ground state energy in terms of a, b, V_0, and physical and mathematical constants. The perturbation parameter is $\frac{a}{b}$. You may use the approximation $\int_{-\epsilon}^{+\epsilon} \cos^2 z \, dz \cong 2\epsilon$ for $\epsilon \ll 1$

6. A 10-eV electron beam moving in the +x direction is incident upon a potential step at $x = 0$: the potential energy for $x < 0$ is 5 eV and the potential energy for $x > 0$ is 0 eV. What fraction of the electrons are reflected from the step?

Do problems from four of five subsections (choose from either the astrophysics subsection or the electronics subsection.) Do 6 of the 8 problems of these sections. Use a different answer book for each subsection.

A. Special Relativity

1. A proton (rest energy 940 MeV) of kinetic energy K strikes a stationary proton. The collision produces a new particle, a pion of rest mass 140 MeV. The process is $pp \to pp\pi$. What is the minimum value of K for this reaction to occur, consistent with conservation of energy and momentum?

2. A car is 5m long when at rest. When the car is moving, length contraction will make it shorter according to a stationary observer. How fast must this car move in order to fit inside a garage of length 4m (according to the same stationary observer)? From the point of view of someone in the car, it is the garage which is moving and therefore contracted. From this perspective, the car will NOT fit inside the garage. How do you explain this apparent paradox?

B. Themal Physics

1. One mole of a monatomic ideal gas is allowed to expand isothermally at 400 K from an initial volume of 10 liters to a final volume of 15 liters.

   a) Find the work done in this process.

   b) This work is used to drive a thermodynamic refrigerator operating between reservoirs of temperatures 200 K and 300 K. What is the maximum amount of heat withdrawn from the low temperature reservoir?

   (The gas constant, $R = 8.314$ J/K)

2. Consider a system of N noninteracting spins in a uniform magnetic field H and temperature T. Each spin can be in one of two states with energies: μH if the spin is up and $-\mu H$ if the spin is down. Find expressions (in terms of T and H) for

   (a) fraction of up spins and

   (b) heat capacity per spin.

C. Optics

1. A 1 cm tall object is placed 5 cm to the left of a first converging lens ($f_1 = 3$ cm) which is followed by a second converging lens ($f_2 = 5$ cm) 5 cm to the right of the first lens.

   a) Draw a ray diagram to find the final image.

   b) Calculate the image position and the magnification.

2. An electron beam passes through a thin film of a crystalline material. The atoms in the material are arranged in a square lattice with an atom-atom spacing, measured along a side of the square, of 0.2 nm. The first order diffraction peak appears 1 cm away from the direct beam, observed on a screen 10 cm behind the thin film. Calculate the wavelength of the electrons.

D. Astrophysics

1. a. Define both monochromatic and integrated flux. If we assume that a star radiates like a black body give an expression for its surface flux of radiation. Now make a simple modification to this expression that allows for the fact that it is not a black body (really just a redefinition). Define luminosity of a star and express it quantitatively in terms of the flux.

   b. The star Sirius has a measured flux at the Earth of $F = 1.23 \times 10^{-4}$cgs. If the parallax is $p = 0.38$ arcsec, determine its luminosity and estimate its radius.

   c. Assuming that Sirius is a black body and that its emitted energy peaks near a frequency of $\nu \sim 1.0 \times 10^{14}$ Hz, estimate its surface temperature. (No, you do not need any constants here. Just think about the rough properties of the Sun.)

   d. Sirius shows a proper motion $\mu = 1.32$ arcsec/yr and the Hβ absorption line in its spectrum has a measured central wavelength of $\lambda = 4861.17$ Å. Derive the space velocity of Sirius.

2. a. At what rate must a supermassive black hole consume mass in order to account for the luminosity (10^{40} W) of a typical quasar? Neglect relativistic corrections.

   b. Note that the above value is independent of the black hole mass. Why then is a supermassive BH needed in a quasar?

   c. The Eddington luminosity is the luminosity where the radiation pressure just balances the gravitational force. Derive an expression for the radiation force and gravitational force on a particle near a quasar. Remember the definition of the momentum of a photon and assume that only one kind of particle in an ionized gas is affected by radiation pressure. Derive an expression for the Eddington mass that corresponds to the Eddington luminosity.

E. Electronics

1. a) Calculate the peak current in a solenoid of resistance $R = 10$ ohms and inductance $L = 0.159$ microhenrys driven by a sinusoidal source at 1000 Hz with a peak voltage of 200 V.

   b) Explain how a capacitor can be inserted appropriately in the circuit to maximize the current.

   c) Calculate its value.

2. a) Calculate the current in the 16 ohm resistor in the circuit shown below.

   b) Determine the voltage measured from A to B.