QUALIFYING EXAMINATION
JANUARY 2001

General Instructions: No reference materials (except for the use of a calculator) are permitted. Do all your work in the answer booklet. Turn in the questions for each part with the answer booklet. There are 90 minutes allotted for each part, with a 30-minute break in between.

PART I: Classical Mechanics

Work any 5 of the 6 problems.

1. **Consider a uniform density flexible cable of mass M and length L that is sitting at rest with half its length hanging over each side of a frictionless pulley (having negligible mass and radius). At time $t = 0$ the cable is given an infinitesimal push to accelerate it.**

a) Write down the equation of motion for the cable.

b) Find the potential energy of the system when one end is at a distance x ($L/2 < x < L$) below the pulley, assuming that the potential energy of the original position is zero.

c) Find the time for the cable to travel the final L/4 of its length before the other end flies off the pulley.

2. **According to Yukawa's theory of nuclear forces, the attractive force between a neutron and a proton inside the nucleus is represented by a potential of the form**

$$V(r) = \frac{ke^{-ar}}{r}$$

where k and a are constants and $k < 0$ and $a > 0$.

a) Find the force F(r) and sketch it.

b) Find the effective radial potential $V_{eff}(r)$ for a given angular momentum L and sketch it.

c) Find the value of the energy E and angular momentum L such that the motion is a circle of radius r_0.

3. **A mass m is thrown upward from the ground with an initial speed $v_z = v_0$. Assume that the air exerts a frictional force proportional to the instantaneous speed.**

a) Write down the equation of motion, and find v (t) and z (t) for the upward motion.

b) Find the time necessary for the mass to reach the maximum height, t_m, and the maximum height $z_m = z(t_m)$ reached.

4. Consider a solid hemisphere of radius R whose density varies linearly with the distance r measured from the origin situated at the center of the base; that is $\rho(r) = \rho_0\, r/R$, where ρ_0 is a constant.

a) Find the mass M and the center of mass of the hemisphere; express the center of mass in terms of R.

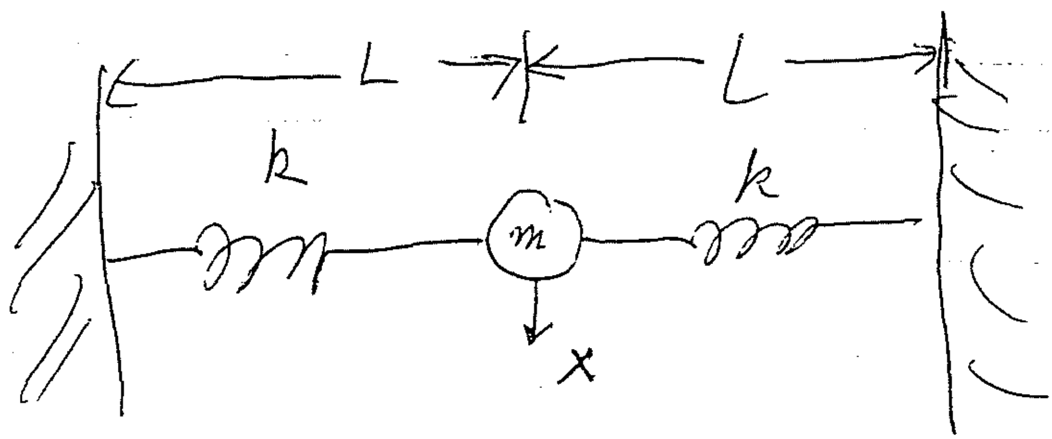
b) Find the moment of inertia I_{zz} of the hemisphere about the axis of symmetry z through the origin; express your answer in terms of M and R.

5. A billiard ball sliding on a frictionless table strikes an identical ball at rest. The balls leave the collision at angles $\pm\,\theta$ with respect to the original direction of motion. Show that after the collision the ratio of rotational kinetic energy of the balls to the initial translational kinetic energy is given by $(1 - {}^{1}/_{2}\cos^{-2}\theta)$, assuming that no energy is dissipated by friction.

6. Consider a mass m suspended between 2 walls by identical springs having force constant k. The mass is displaced by a small distance x as shown and released. Ignoring gravity:

a) Write down the exact T and V, where T is the kinetic energy and V is the potential energy.

b) For x < < L, find the approximate Lagrangian and write down the corresponding equation of motion.



PART II: **Classical Electromagnetism** (work any 5 of the 6 problems)

1. Two protons are separated by a distance 2d. An electron lies half way between. If it is displaced an infinitesimal distance perpendicular to the line between the two protons, with what angular frequency will it oscillate?

2. A wire shaped in the form of an equilateral triangle of side length L is subjected to a spatially uniform, perpendicular B field increasing with time as $B(t) = B_0\,(t/t_0)^{1/2}$. What current is induced in the wire if its resistance is R?

3. The electrostatic potential in a spherical region of radius R is given by $\Phi\,(r, \theta) = V\,(R/r)^2 \cos(\theta)$. What is the charge density in the region?

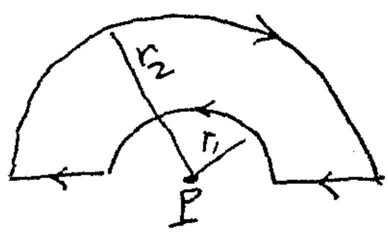
4. An axially symmetric charge density everywhere in space is of the form

$$\rho(r, \theta, z) = \rho_0 \frac{b}{r} e^{-\frac{r}{b}}$$

where the coordinate r is the distance from the z axis. What is the electric field everywhere in space?

5. What is the capacitance of a system of two concentric hollow spherical conductors of radii r_1 and r_2 separated by vacuum? Each conductor has infinitesimal thickness, and $r_1 > r_2$.

6. A wire carrying current I takes the form of two semi-circular lengths of radii r_1 and r_2 joined by straight sections as shown. What is the magnitude of the B field at the origin, P?



PART III: Quantum Mechanics (work any 5 of the 6 problems)

1. A positron is a particle with the same mass as an electron but with opposite charge. Electrons and positrons can form bound states called positronium in which the electron and positron orbit about a common center of mass. Assuming the particles travel in circular orbits, use the Bohr quantization condition on the angular momentum to calculate the energy levels for positronium. What is the ground state energy in eV?

2. At t = 0, a particle in a harmonic oscillator potential is in the initial state

$$|\psi(0)>= \frac{1}{\sqrt{5}}|E_1> + \frac{2}{\sqrt{5}}|E_2> .$$

The E_i's are the energy eigenstates, where $|E_1>$ is the first excited state and $|E_2>$ is the second.

a) What is the expectation value of the energy in the state $|\psi(0)>$?

b) Find $|\psi(t)>$. Is this a stationary state?

c) For the position operator x, evaluate the expectation value $<\psi(t)|\ x\ |\psi(t)>$. What is the frequency of oscillation of this expectation value? [Hint: Use the functional form of the energy eigenfunctions and the recursion relation given at the end of the exam to evaluate the expectation value].

3. For a hydrogen atom in the maximum angular momentum state the wave function is given by

$$\psi_{n,n-1,m}(r,\theta,\phi) = N_n r^{n-1} e^{-r/na} Y^m_{n-1}(\theta,\phi)$$

where

$$N_n = \sqrt{\frac{2^{2n+1}}{(na)^{2n+1}(2n)!}},$$

the Y_l^m's are the spherical harmonics and a is the first Bohr radius.

a) Calculate $< 1/r >$ for a hydrogen atom in this state.

b) Using the result from part a) calculate the expectation value for the hydrogen atom potential energy.

c) Calculate the kinetic energy for a hydrogen atom in this state. [Hint: The virial theorem is valid for this system.]

4. **a) Find the eigenvectors of**

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

the y-component of the spin operator. Express them as spinors.

b) Suppose that an electron is in the spin state

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

If we measure the y component of the spin, what is the probability for finding a value of

$$+\tfrac{1}{2}\hbar?$$

5. **Suppose that five electrons are placed in a one-dimensional infinite potential well of length L. What is the energy of the ground state of this system of five electrons? What is the net spin of the ground state? Take the exclusion principle into account, and ignore the Coulomb interaction of the electrons with each other.**

6. **Suppose that the electron in a hydrogen atom is perturbed by a repulsive potential concentrated at the origin. Assume the potential has the form of a 3-dimensional delta function, so the perturbed Hamiltonian is**

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} + A\delta(\mathbf{r})$$

where A is a constant.

a) To first order in A, find the change in the energy of the state with principal and angular momentum quantum numbers $n = 1, l = 0$. [Hint: $\psi_{n00}(0) = 2/(\sqrt{4\pi}(na_0)^{3/2})$; where a_0 is the first Bohr radius.]

c) Find the change in the wave function.

Useful Integral

$$\int_0^\infty r^m e^{-\alpha r} dr = m!\, \alpha^{-m-1}$$

Harmonic Oscillator Wave Functions

$$\psi_n = A_n e^{-\xi^2/2} H_n(\xi)$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$A_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!(\hbar\omega)^n}}$$

Recursion Relation for Hermite Polynomials

$$\xi H_1(\xi) = \frac{H_2(\xi) + 2H_0(\xi)}{2}$$

PART IV: Mixed Topics

Do problems from 4 of the 5 sections (choose either the astrophysics or electronics section). Do 6 of the 8 problems in these sections, with at least one from each of the 4 sections. Use a different answer book for each lettered section.

A. **Relativity**

1. a) A reference frame F' is moving with velocity V in the x' direction with respect to a frame F, which has mutually orthogonal x, y, z directions parallel to the x', y', z' directions of frame F'. From the Lorentz transformations of x', y', z', t' in terms of x, y, z, t, V, derive the velocity transformations of Vx', Vy', Vz' in terms of Vx, Vy, Vz, V.

   b) An alien space ship approaches the Earth with a velocity 70% the speed of light c. The space ship fires a proton beam at the Earth with speed 0.9c with respect to the space ship's reference frame. What is the velocity of the proton beam from the Earth's point of view?

2. A particle of mass M_o at rest decays into a photon and a particle with rest mass m_o.

   a) What are the energy of the photon and the velocity of the massive particle in terms of M_o and m_o?

   b) Solve for the specific case of $m_o = M_o/2$.

PART IV:

B. Thermo

1. A monoatomic gas fills a volume of 10^{-3} m^3 at $T = 300K$ and $p = 10^5$ Pa. In a reversible, isothermal process the gas expands against a constant external pressure while 100J of thermal energy is transferred to the gas through contact with a heat reservoir.

   a) What is the final volume of the gas?

   b) Does the entropy of the gas change during the expansion?

   c) If the expansion was adiabatic - would the entropy change in this case?

2. A Pt atom in a catalytic converter can either be 'clean' (nothing absorbed), can have absorbed a NO molecule or can have absorbed a CO molecule. The absorption energies for a molecule of NO or CO are respectively $\varepsilon_{NO} = 0.1$ eV and $\varepsilon_{CO} = 0.2$ eV.

   a) Write down the Gibbs sum for this system.

   b) Which ratio of absolute activities $\lambda_{CO} / \lambda_{NO}$ is necessary in order to maintain equal absorption probability for NO and CO at 600°C?

   c) What would (in case b) be their ratio of concentrations?

PART IV:

C. OPTICS

1. (Note: Either use MKS units or specify the system of units that you choose to employ for this problem.) A plane electromagnetic wave of angular frequency ω travels in empty space along the negative y axis of a right-handed x - y - z coordinate system. The magnitude of the electric field of this wave is E_0. Assume that the wave is plane polarized and that its electric field vibrates in the x direction.

    a) Write an expression describing the (vector) electric field as a function of space and time, $E(x, y, z, t)$.

    b) Write an expression describing the (vector) magnetic field of this wave as a function of space and time, $B(x, y, z, t)$.

    c) Write an expression for the electric field of a similar wave traveling (in the same direction with the same amplitude) in a material medium whose (optical) dielectric constant is ε.

    d) Write an expression for the magnetic field of a similar wave traveling (in the same direction with the same amplitude) in a material medium whose (optical) dielectric constant is ε.

2. A single slit is 5×10^{-3} mm wide. Monochromatic light of wavelength 500 nm is normally incident on this slit. This light then hits a white screen 2 meters away from the slit.

    a) How far (in cm) on the screen from the central maximum is the first minimum in intensity?

    b) Is the intensity at this minimum really zero, or only small compared to the maximum intensity?

    c) Approximately where is the next maximum in intensity?

    d) Is the actual maximum closer to the central maximum or farther away than your answer in c?

    e) Compared to the intensity of the central maximum, roughly estimate the intensity of the maximum in part c. Give your answer as a percent.

    f) If we increase the wavelength of the light what happens to the interference pattern observed on the screen?

PART IV:

D. Astrophysics

1. For a circular orbit, consider the determination of an extrasolar planet's mass from the estimated stellar mass and velocity amplitude of the perturbed motion of the star alone. Derive how the planetary mass depends on the (usually unknown) angle between the line of sight and the orbital plane.

2. Quasar luminosities are found to reach 10^{47} erg/second. Estimate the mass consumption rate to power these objects if the energy comes from:

   a) fusion

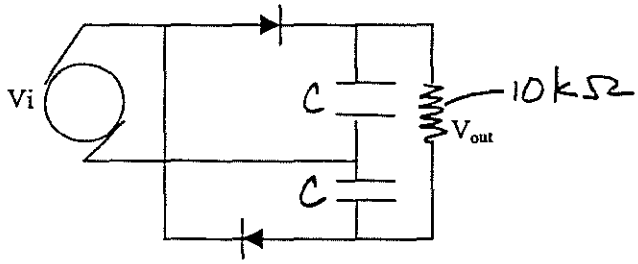
   b) gravitational release

   c) annihilation.

PART IV:

E. Electronics

1. Consider the circuit shown below:



The two capacitors have the same value. The input AC voltage is $V_I = 141\sin(377t)$.

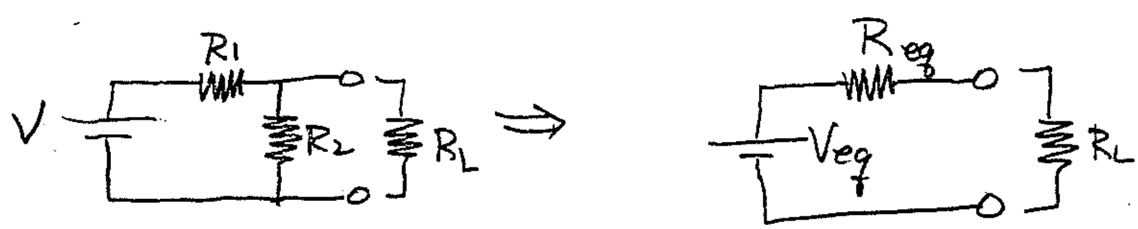
a) What does this circuit do?

b) Assume the capacitors are electrolytic and assign + and - signs to show the correct polarity for the capacitors.

c) What is the maximum DC voltage that could be expected across the resistor?

d) What is the minimum value of each capacitor necessary to be sure the voltage drop between charging pulses is less than 10% of the output voltage?

2. Consider the circuit shown below:



a) Determine the values of R_{eq} and V_{eq} for the Thevenin equivalent circuit.

b) Use the results of a. to show that the maximum power transfer will occur when $R_L = R_{eq}$.